

Topic 4 · Sheet A: Questions

Ordered pairs, cartesian products, lists and strings

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 3

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need ordered pairs, cartesian products, relations, graphical relations, lists and strings, plus Topics 1 to 3.

Notation. $X \times Y$ is the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$, and $X^2 = X \times X$. A relation from X to Y is a subset of $X \times Y$; it is graphical when every $x \in X$ is paired with exactly one y . A list of length n taken from A is a function $\llbracket n \rrbracket \rightarrow A$, written $(a_0, a_1, \dots, a_{n-1})$. A string over an alphabet is a list of its letters; binary strings are strings over $\{0, 1\}$.

1

4 marks

- (a) Is $(1, 2) = (2, 1)$? [1]
- (b) Is $\{1, 2\} = \{2, 1\}$? [1]
- (c) Find all real numbers x and y with $(x + 1, 2y) = (4, 6)$. [1]
- (d) How many elements does $\{(0, 1), (1, 0), (0, 1)\}$ have? [1]

2

5 marks

- (a) List the elements of $\llbracket 2 \rrbracket \times \llbracket 3 \rrbracket$. [2]
- (b) Find $|\llbracket 3 \rrbracket \times \llbracket 2 \rrbracket|$. Is $\llbracket 2 \rrbracket \times \llbracket 3 \rrbracket = \llbracket 3 \rrbracket \times \llbracket 2 \rrbracket$? Explain. [2]
- (c) Write down $\llbracket 3 \rrbracket \times \emptyset$. [1]

3

5 marks

Find the number of elements of each set.

- (a) $\llbracket 4 \rrbracket^2$ [1]
- (b) $\mathcal{P}(\llbracket 2 \rrbracket) \times \llbracket 3 \rrbracket$ [1]
- (c) $\{(x, y) \in \llbracket 4 \rrbracket^2 \mid x < y\}$ [1]
- (d) $\{(x, y) \in \llbracket 4 \rrbracket^2 \mid x = y\}$ [1]

(e) $\{(x, y) \in \mathbb{[4]}^2 \mid x + y = 3\}$ [1]

4

6 marks

- (a) List the relation $R = \{(x, y) \in \mathbb{[3]}^2 \mid x \leq y\}$ on $\mathbb{[3]}$. [2]
 (b) Is R graphical? Explain. [1]
 (c) List the identity relation on $\mathbb{[3]}$. Is it graphical? [2]
 (d) How many ordered pairs are in the universal relation on $\mathbb{[3]}$? [1]

5

6 marks

For each relation from $\mathbb{[3]}$ to $\mathbb{[3]}$, decide whether it is graphical. When it is, give the function it defines as a table of values.

- (a) $G_1 = \{(0, 1), (1, 1), (2, 0)\}$ [2]
 (b) $G_2 = \{(0, 0), (1, 2)\}$ [1]
 (c) $G_3 = \{(0, 1), (0, 2), (1, 0), (2, 2)\}$ [1]
 (d) $G_4 = \{(x, y) \in \mathbb{[3]}^2 \mid x + y = 2\}$ [2]

6

4 marks

Decide whether each relation on $\mathbb{R}$ is graphical, with a brief reason.

- (a) $\{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$ [1]
 (b) $\{(x, y) \in \mathbb{R}^2 \mid x = y^3\}$ [1]
 (c) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ [1]
 (d) $\{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2\}$ [1]

7

6 marks

- (a) How many relations are there on $\mathbb{[2]}$? [2]
 (b) How many relations are there from $\mathbb{[2]}$ to $\mathbb{[3]}$? [1]
 (c) How many of the relations in (b) are graphical? [1]
 (d) How many relations on $\mathbb{[3]}$ contain the identity relation? [2]

8

6 marks

- (a) Are the lists $(0, 1, 2)$ and $(0, 1, 2, 2)$ equal? Why? [1]
 (b) How many lists of length 3 can be taken from $\mathbb{[2]}$? [1]
 (c) Let $L: \mathbb{[4]} \rightarrow \mathbb{N}$ be the list with $L(k) = k^2$. Write L in the notation (a_0, a_1, a_2, a_3) . [1]
 (d) Write out the list $(a_k)_{k=0}^4$ with $a_k = 3k + 1$. [1]
 (e) How many lists of length 3, taken from $\mathbb{[4]}$, have three different entries? [2]

9

7 marks

- (a) How many strings of length 2 are there over the alphabet $\{a, b, c\}$? [1]
- (b) List all binary strings of length 2. [1]
- (c) How many binary strings of length 6 are there? [1]
- (d) How many binary strings of length 6 have exactly two bits equal to 1? Explain your count. [3]
- (e) How many binary strings of length 5 start and end with the same bit? [1]

10

7 marks

Let $A = \{a, b, c\}$.

- (a) How many strings over A have length at most 2? Remember the empty string. [2]
- (b) How many strings of length 4 over A have no two neighbouring letters equal? Explain your count. [3]
- (c) A palindrome is a string that reads the same backwards. How many strings of length 3 over A are palindromes? [2]

11

9 marks

Exam style. Let $X = \llbracket 2 \rrbracket$ and $Y = \{a, b\}$.

- (a) List the elements of $X \times Y$. [1]
- (b) How many relations are there from X to Y ? [1]
- (c) List every graphical relation from X to Y , as a set of ordered pairs. [2]
- (d) Which of these define bijections? Explain why the others do not. [3]
- (e) In the Kuratowski construction of ordered pairs, $(x, y) = \{\{x\}, \{x, y\}\}$. Write $(0, 1)$ and $(1, 1)$ this way, and say how many elements each has. [2]

12

9 marks

Exam style.

- (a) How many lists of length 3 can be taken from $\llbracket 5 \rrbracket$? [1]
- (b) How many of them are strictly increasing, that is $a_0 < a_1 < a_2$? [2]
- (c) How many subsets of $\llbracket 5 \rrbracket$ have exactly 3 elements? Explain why the answer matches (b). [2]
- (d) How many binary strings of length 5 have exactly three bits equal to 1? Describe a bijection between these strings and the subsets in (c), and its inverse. [4]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.