

Topic 4 · Sheet A: Worked answers and mark scheme

Ordered pairs, cartesian products, lists and strings

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 3

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Ordered and unordered pairs

4 marks

(a) **No**: the first entries differ. (b) **Yes**: sets ignore order. (c) $x = 3, y = 3$: matching entries gives $x + 1 = 4$ and $2y = 6$. (d) **2**: $(0, 1)$ is listed twice, and $(0, 1) \neq (1, 0)$.

B1	(a)	1
B1	(b)	1
B1	(c) both values	1
B1	(d)	1

Why this question Ordered pairs are equal exactly when their first entries agree and their second entries agree. Part (c) uses this as a pair of equations.

2 Listing a cartesian product

5 marks

(a) $(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)$.

(b) $|\mathbb{[3]} \times \mathbb{[2]}| = 6$. **The sets are not equal**: $(0, 2) \in \mathbb{[2]} \times \mathbb{[3]}$ but $(0, 2) \notin \mathbb{[3]} \times \mathbb{[2]}$, since $2 \notin \mathbb{[2]}$.

(c) $\emptyset$: there is no possible second entry.

B2	(a) all six. <i>B1 for four or five correct, no wrong entries</i>	2
B1	(b) 6	1
B1	(b) not equal, with an element in one but not the other	1
B1	(c) $\emptyset$	1

Common mistake Saying the two products in (b) are equal because they have the same size. Equal size means same cardinality, not equality.

3 Sizes of subsets of products

5 marks

- (a) **16**.
- (b) $4 \times 3 = \mathbf{12}$.
- (c) **6**: $(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)$.
- (d) **4**: the pairs (x, x) .
- (e) **4**: $(0, 3), (1, 2), (2, 1), (3, 0)$.

B1	each part correct	5
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Why this question Parts (c) and (d) split the 16 pairs: 6 with $x < y$, 4 with $x = y$, and so 6 with $x > y$. Spotting that symmetry saves listing.

4 Relations as sets of pairs

6 marks

- (a) $R = \{(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 2)\}$.
- (b) **No**: 0 is related to three different elements.
- (c) $\{(0, 0), (1, 1), (2, 2)\}$, and **it is graphical**: each x is paired with exactly one y , namely itself. It is the relation of $\text{Id}_{\llbracket 3 \rrbracket}$.
- (d) **9**: the universal relation is all of $\llbracket 3 \rrbracket^2$.

B2	(a) all six pairs. <i>B1 for five correct, no wrong entries</i>	2
B1	(b) no, with reason	1
B1	(c) the identity relation	1
B1	(c) graphical	1
B1	(d) 9	1

Why this question The definition makes a relation nothing more than a set of pairs. Once you see that, "is it graphical?" is a question you can check pair by pair.

5 Graphical or not?

6 marks

- (a) **Graphical**. The function is $f(0) = 1, f(1) = 1, f(2) = 0$.
- (b) **Not graphical**: 2 is not paired with anything.
- (c) **Not graphical**: 0 is paired with both 1 and 2.
- (d) $G_4 = \{(0, 2), (1, 1), (2, 0)\}$. **Graphical**, with $f(x) = 2 - x$: $f(0) = 2, f(1) = 1, f(2) = 0$.

B1	(a) graphical	1
B1	(a) table	1
B1	(b) not graphical, naming 2	1
B1	(c) not graphical, naming 0	1
B1	(d) graphical	1
B1	(d) table	1

Why this question A relation can fail to be graphical in two ways: some x has no partner, as in (b), or some x has two, as in (c). Both break “exactly one”.

6 The vertical line test

4 marks

- (a) **Graphical**: each x gives one $y = x^3$.
- (b) **Graphical**: each x gives one $y = \sqrt[3]{x}$, since every real number has exactly one real cube root.
- (c) **Not graphical**: $x = 0$ gives $y = \pm 1$ (and $x = 2$ gives no y).
- (d) **Not graphical**: $x = 1$ gives $y = 1$ and $y = -1$.

B1	each part, with a valid reason	4
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Common mistake Part (b) looks like the sideways parabola $x = y^2$, which is not graphical. But cubing, unlike squaring, is a bijection on $\mathbb{R}$, so each vertical line meets this curve exactly once.

7 Counting relations

6 marks

- (a) $[2]^2$ has 4 elements, and a relation is any subset of it: $2^4 = 16$.
- (b) $[2] \times [3]$ has 6 elements: $2^6 = 64$.
- (c) Graphical relations are functions $[2] \rightarrow [3]$: $3^2 = 9$.
- (d) The 3 identity pairs must be in; the other $9 - 3 = 6$ pairs are free: $2^6 = 64$.

M1	(a) relations are subsets of $\llbracket 2 \rrbracket^2$	1
A1	(a) 16	1
B1	(b) 64	1
B1	(c) 9	1
M1	(d) fixing the identity pairs and counting the free ones	1
A1	(d) 64	1

Why this question This combines three earlier ideas: a relation is a subset (so count with 2^n), a function is a graphical relation (so count with n^m), and fixed elements leave the rest free (Topic 1).

8 Lists

6 marks

- (a) **No**: they have different lengths, 3 and 4.
- (b) $2^3 = 8$: a list is a function $\llbracket 3 \rrbracket \rightarrow \llbracket 2 \rrbracket$.
- (c) $(0, 1, 4, 9)$.
- (d) $(1, 4, 7, 10, 13)$.
- (e) $4 \times 3 \times 2 = 24$.

B1	(a)	1
B1	(b) 8	1
B1	(c)	1
B1	(d)	1
M1	(e) $4 \times 3 \times 2$ or equivalent	1
A1	(e) 24	1

Why this question Lists keep order and repeats, which is exactly what sets throw away. Counting lists is counting functions, as in Topic 2.

9 Strings

7 marks

- (a) $3^2 = 9$.
- (b) 00, 01, 10, 11.
- (c) $2^6 = 64$.
- (d) Choose the two positions for the 1s: **15**. (Position pairs from $\llbracket 6 \rrbracket$: $5 + 4 + 3 + 2 + 1$.)
- (e) First bit: 2 choices; the last bit is then fixed; the middle three bits are free: $2 \times 2^3 = 16$.

B1	(a) 9	1
B1	(b) all four	1
B1	(c) 64	1
M1	(d) choosing positions of the two 1s	1
M1	(d) a systematic count, e.g. $5 + 4 + 3 + 2 + 1$	1
A1	(d) 15	1
B1	(e) 16	1

Why this question Part (d) uses a key counting idea: a string with a given number of 1s is decided by *where* the 1s go.

10 Counting strings with conditions

7 marks

- (a) One string of length 0, 3 of length 1 and 9 of length 2: $1 + 3 + 9 = 13$.
- (b) The first letter has 3 choices; each later letter must differ from the one before, so 2 choices each: $3 \times 2 \times 2 \times 2 = 24$.
- (c) The first letter decides the last; the middle letter is free: $3 \times 3 = 9$.

M1	(a) adding counts for lengths 0, 1 and 2	1
A1	(a) 13, including the empty string	1
M1	(b) 3 choices for the first letter	1
M1	(b) 2 choices for each later letter	1
A1	(b) 24	1
M1	(c) noticing the last letter is fixed by the first	1
A1	(c) 9	1

Common mistake In (a), forgetting the empty string ϵ_A , giving 12.

11 Exam style: relations and pairs as sets

9 marks

- (a) $(0, a), (0, b), (1, a), (1, b)$.
- (b) $2^4 = 16$.
- (c) $\{(0, a), (1, a)\}, \{(0, a), (1, b)\}, \{(0, b), (1, a)\}, \{(0, b), (1, b)\}$.
- (d) $\{(0, a), (1, b)\}$ **and** $\{(0, b), (1, a)\}$: the other two send both elements to the same letter.
- (e) $(0, 1) = \{\{0\}, \{0, 1\}\}$, **with 2 elements**. $(1, 1) = \{\{1\}, \{1, 1\}\} = \{\{1\}\}$, **with 1 element**, since $\{1, 1\} = \{1\}$.

B1	(a)	1
B1	(b) 16	1
B2	(c) all four. <i>B1 for three, no wrong entries</i>	2
B2	(d) both, and no others. <i>B1 for one</i>	2
E1	(d) the others send both elements to the same letter	1
B1	(e) $(0, 1)$ with 2 elements	1
B1	(e) $(1, 1)$ simplified, with 1 element	1

Why this question Part (e) shows how an ordered pair can be built from sets alone. Notice that the pair $(1, 1)$ collapses to a one-element set; the construction still works, because $\{\{x\}\}$ can only come from the pair (x, x) .

12 Exam style: lists, subsets and strings

9 marks

- (a) $5^3 = 125$.
- (b) **10**. An increasing list is decided by its set of three entries, written in order.
- (c) **10**: $\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 4\}, \{0, 3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$. Each subset gives exactly one increasing list, by writing its elements in order, and every increasing list arises this way.
- (d) **10**. Send a string to the set of positions holding a 1; for example $10110 \mapsto \{0, 2, 3\}$. The inverse puts a 1 in each listed position and 0 elsewhere.

B1	(a) 125	1
M1	(b) linking increasing lists to subsets, or a full list	1
A1	(b) 10	1
B1	(c) 10	1
E1	(c) the matching explained	1
B1	(d) 10	1
M1	(d) positions of the 1s	1
A1	(d) bijection clearly described	1
B1	(d) the inverse described	1

Why this question Three different-looking objects, increasing lists, 3-element subsets and binary strings with three 1s, are all counted by the same number. Finding the bijections between them is often easier than counting each one directly.

Total: 74 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.