

Topic 3 · Sheet A: Questions

Cardinality, Cantor's theorem and Cantor–Schröder–Bernstein

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 2

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need cardinality, finite and infinite sets, Cantor's theorem and the Cantor–Schröder–Bernstein theorem, plus Topics 1 and 2.

Notation. $|X| = |Y|$ means there is a bijection $X \rightarrow Y$, and $|X| = n$ means there is a bijection $\llbracket n \rrbracket \rightarrow X$. Where a question asks for a bijection, give its domain, codomain and rule. Where a, b, c, d appear, they are different objects, none of which is a set.

1

4 marks

Find $|X|$ for each set X .

- (a) $X = \{k \in \mathbb{N} \mid k^2 < 30\}$ [1]
- (b) $X = \mathcal{P}(\llbracket 3 \rrbracket)$ [1]
- (c) $X = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ [1]
- (d) $X = \{k \in \llbracket 20 \rrbracket \mid k \text{ is a multiple of } 4\}$ [1]

2

4 marks

Write down a bijection in each case.

- (a) $\llbracket 3 \rrbracket \rightarrow \{a, b, c\}$ [1]
- (b) $\llbracket 4 \rrbracket \rightarrow \{k \in \mathbb{N} \mid 10 \leq k \leq 13\}$ [1]
- (c) $\llbracket 5 \rrbracket \rightarrow \{0, 2, 4, 6, 8\}$ [1]
- (d) Write down the inverse of your bijection in (c). [1]

3

5 marks

Decide whether the two sets have the same cardinality. Justify each answer with a bijection or a short reason.

- (a) $\{1, 2, 3\}$ and $\{\emptyset, \{\emptyset\}, \llbracket 2 \rrbracket\}$ [1]
- (b) $\llbracket 4 \rrbracket$ and $\mathcal{P}(\llbracket 2 \rrbracket)$ [1]

(c) $\llbracket 3 \rrbracket$ and $\mathcal{P}(\llbracket 3 \rrbracket)$ [1]

(d) $\mathbb{N}$ and $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$ [2]

4

6 marks

Find a bijection between each pair of sets, and give its inverse.

(a) $\mathbb{N}$ and $\{n \in \mathbb{N} \mid n \geq 5\}$ [2]

(b) $\mathbb{N}$ and $\{n \in \mathbb{N} \mid n \text{ is odd}\}$ [2]

(c) $\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\{x \in \mathbb{R} \mid 0 < x < 2\}$ [2]

5

6 marks

This question follows the proof of Cantor's theorem for one particular function. Let $f: \llbracket 3 \rrbracket \rightarrow \mathcal{P}(\llbracket 3 \rrbracket)$ be given by

$$f(0) = \{1, 2\}, \quad f(1) = \{1\}, \quad f(2) = \emptyset.$$

(a) For each $x \in \llbracket 3 \rrbracket$, decide whether $x \in f(x)$. [1]

(b) Find the set $C = \{x \in \llbracket 3 \rrbracket \mid x \notin f(x)\}$. [2]

(c) Check that C is not equal to any of $f(0)$, $f(1)$, $f(2)$. [1]

(d) For each x , name an element that lies in exactly one of C and $f(x)$. Explain why this means C can never be a value of f . [2]

6

6 marks

Let $f, g: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$ be given by $f(n) = \{k \in \mathbb{N} \mid k \geq n\}$ and $g(n) = \llbracket n \rrbracket$.

(a) Write down $f(0)$ and $f(3)$. [1]

(b) Find $C_f = \{n \in \mathbb{N} \mid n \notin f(n)\}$. [2]

(c) Is C_f a value of f ? Explain. [2]

(d) Find $C_g = \{n \in \mathbb{N} \mid n \notin g(n)\}$, and say whether it is a value of g . [1]

7

5 marks

(a) How many bijections are there from $\{a, b, c, d\}$ to $\llbracket 4 \rrbracket$? [1]

(b) How many injections are there from $\mathcal{P}(\llbracket 1 \rrbracket)$ to $\llbracket 3 \rrbracket$? [2]

(c) How many bijections are there from $\mathcal{P}(\llbracket 2 \rrbracket)$ to $\llbracket 4 \rrbracket$? [1]

(d) How many bijections are there from $\llbracket 3 \rrbracket$ to $\llbracket 4 \rrbracket$? [1]

8

6 marks

Let $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$.

(a) Let $f: \mathbb{N} \rightarrow E$ be given by $f(n) = 4n$. Is f injective? Is it surjective? If not, name a value it misses. [2]

- (b) Let $g: E \rightarrow \mathbb{N}$ be given by $g(m) = m$. Is g injective? Is it surjective? If not, name a value it misses. [2]
- (c) So there are injections both ways, and the Cantor–Schröder–Bernstein theorem says there is a bijection $\mathbb{N} \rightarrow E$. Write one down, with its inverse. [2]

9

6 marks

Decide whether each set is finite or infinite. For each finite set, give its cardinality.

- (a) $\mathcal{P}(\llbracket 10 \rrbracket)$ [1]
- (b) $\{n \in \mathbb{N} \mid n^2 < 1000\}$ [1]
- (c) $\{n \in \mathbb{N} \mid n \text{ is a multiple of } 7\}$ [1]
- (d) $\{x \in \mathbb{R} \mid x^2 = 2\}$ [1]
- (e) $\{x \in \mathbb{Q} \mid x^2 = 2\}$ [1]
- (f) $\mathcal{P}(\mathbb{N})$ [1]

10

7 marks

This question explains why $|\mathcal{P}(\llbracket n \rrbracket)| = 2^n$. Let F be the set of all functions $\llbracket 3 \rrbracket \rightarrow \llbracket 2 \rrbracket$.

- (a) Find $|F|$ and $|\mathcal{P}(\llbracket 3 \rrbracket)|$. [1]
- (b) Describe a bijection $\Phi: F \rightarrow \mathcal{P}(\llbracket 3 \rrbracket)$, and describe its inverse. [3]
- (c) Under your Φ , which subset corresponds to the function with $f(0) = 1, f(1) = 0, f(2) = 1$? Which function corresponds to $\{1\}$? [2]
- (d) Explain in one sentence why the same idea gives $|\mathcal{P}(\llbracket n \rrbracket)| = 2^n$. [1]

11

10 marks

Exam style. Let $X = \{a, b\}$.

- (a) List the elements of $\mathcal{P}(X)$. [1]
- (b) How many functions are there from X to $\mathcal{P}(X)$? [1]
- (c) Let $f: X \rightarrow \mathcal{P}(X)$ have $f(a) = \{a, b\}$ and $f(b) = \{a\}$. Find $C = \{x \in X \mid x \notin f(x)\}$, and check that C is not a value of f . [3]
- (d) How many of the functions from X to $\mathcal{P}(X)$ are surjective? Explain. [2]
- (e) How many of them are injective? Explain your count. [3]

12

9 marks

Exam style. Let $T = \{n \in \mathbb{N} \mid n \text{ is not a multiple of } 3\}$.

- (a) Find a bijection $\mathbb{N} \rightarrow \{n \in \mathbb{N} \mid n \geq 1\}$, and its inverse. [2]
- (b) Let $h: \mathbb{N} \rightarrow T$ be given by

$$h(n) = \begin{cases} 3k+1 & \text{if } n = 2k \text{ is even,} \\ 3k+2 & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

Find $h(0), h(1), \dots, h(5)$. [1]

(c) The function h is a bijection. Write down a rule for $h^{-1}: T \rightarrow \mathbb{N}$, treating the two kinds of element of T separately. Check that $h^{-1}(h(4)) = 4$ and $h^{-1}(h(5)) = 5$. [4]

(d) Find the n with $h(n) = 10$, and the n with $h(n) = 11$. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.