

## Topic 3 · Sheet A: Worked answers and mark scheme

# Cardinality, Cantor's theorem and Cantor–Schröder–Bernstein

Suggested time

**About 75 minutes**

Total marks

**74**

Questions

**12**

Builds on

**Topics 1 to 2**

**How marks are awarded.** Each mark has a code, as on most UK exam papers.

**B** A mark for a correct statement or value on its own. It does not depend on any other mark.

**M** A method mark, for a correct approach, even if the final answer is wrong.

**A** An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

**E** An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

## 1 Cardinality of finite sets

4 marks

(a) **6**: the set is  $\{0, 1, 2, 3, 4, 5\}$ , since  $5^2 = 25 < 30 \leq 36$ . (b) **8**. (c) **3**: the three elements are all different. (d) **5**: the set is  $\{0, 4, 8, 12, 16\}$ .

|           |       |   |
|-----------|-------|---|
| <b>B1</b> | (a) 6 | 1 |
| <b>B1</b> | (b) 8 | 1 |
| <b>B1</b> | (c) 3 | 1 |
| <b>B1</b> | (d) 5 | 1 |

**Common mistake** In (a), forgetting 0. In (c), thinking  $\{\{\emptyset\}\}$  and  $\{\emptyset\}$  are the same: one contains  $\emptyset$  and the other contains  $\{\emptyset\}$ .

## 2 Writing down bijections

4 marks

- (a)  $0 \mapsto a, 1 \mapsto b, 2 \mapsto c$  (any matching of the three elements works). (b)  $k \mapsto k + 10$ . (c)  $k \mapsto 2k$ . (d)  $m \mapsto m/2$ , from  $\{0, 2, 4, 6, 8\}$  to  $\llbracket 5 \rrbracket$ .

|    |                                          |   |
|----|------------------------------------------|---|
| B1 | (a) any bijection                        | 1 |
| B1 | (b)                                      | 1 |
| B1 | (c)                                      | 1 |
| B1 | (d) with the correct domain and codomain | 1 |

**Why this question** By definition,  $|X| = n$  means there is a bijection from  $\llbracket n \rrbracket$  to  $X$ . Writing one down is exactly what “counting” a set means in this module.

## 3 Same cardinality or not?

5 marks

- (a) **Yes.** For example  $1 \mapsto \emptyset, 2 \mapsto \{\emptyset\}, 3 \mapsto \llbracket 2 \rrbracket$ .  
(b) **Yes.** Both have 4 elements; for example  $0 \mapsto \emptyset, 1 \mapsto \{0\}, 2 \mapsto \{1\}, 3 \mapsto \{0, 1\}$ .  
(c) **No.**  $|\llbracket 3 \rrbracket| = 3$  and  $|\mathcal{P}(\llbracket 3 \rrbracket)| = 8$ . (Cantor’s theorem also rules out any bijection.)  
(d) **Yes.**  $n \mapsto 2n$  is a bijection  $\mathbb{N} \rightarrow E$ , with inverse  $m \mapsto m/2$ .

|    |                                                           |   |
|----|-----------------------------------------------------------|---|
| B1 | (a) yes, with a bijection                                 | 1 |
| B1 | (b) yes, with a bijection or both sizes                   | 1 |
| B1 | (c) no, with a reason                                     | 1 |
| B1 | (d) yes                                                   | 1 |
| B1 | (d) the bijection $n \mapsto 2n$ , or another correct one | 1 |

**Why this question** Part (d) is the first surprise of the topic:  $E$  is a proper subset of  $\mathbb{N}$ , yet the two have the same cardinality. That cannot happen for finite sets.

## 4 Bijections between infinite sets

6 marks

- (a)  $n \mapsto n + 5$ , with inverse  $m \mapsto m - 5$ .  
(b)  $n \mapsto 2n + 1$ , with inverse  $m \mapsto (m - 1)/2$ .  
(c)  $x \mapsto 2x$ , with inverse  $y \mapsto y/2$ .

|    |                                             |   |
|----|---------------------------------------------|---|
| B1 | each correct bijection                      | 3 |
| B1 | each correct inverse, with the right domain | 3 |

**Common mistake** Giving a rule whose values land outside the codomain, such as  $n \mapsto 2n$  for part (b). Always check that every value is in the codomain and every element of the codomain is hit.

## 5 Cantor's argument for one function

6 marks

- (a)  $0 \notin f(0), 1 \in f(1), 2 \notin f(2)$ .  
 (b)  $C = \{0, 2\}$ .  
 (c) The values of  $f$  are  $\{1, 2\}$ ,  $\{1\}$  and  $\emptyset$ ; **none of them is  $\{0, 2\}$** .  
 (d)  $x = 0$ :  $0 \in C$  but  $0 \notin f(0)$ .  $x = 1$ :  $1 \in f(1)$  but  $1 \notin C$ .  $x = 2$ :  $2 \in C$  but  $2 \notin f(2)$ . So  $C$  and  $f(x)$  always disagree about the element  $x$  itself, and so  $C \neq f(x)$  for every  $x$ .

|           |                                                            |   |
|-----------|------------------------------------------------------------|---|
| <b>B1</b> | (a) all three correct                                      | 1 |
| <b>M1</b> | (b) using the rule: $x \in C$ exactly when $x \notin f(x)$ | 1 |
| <b>A1</b> | (b) $\{0, 2\}$                                             | 1 |
| <b>B1</b> | (c)                                                        | 1 |
| <b>B1</b> | (d) the element $x$ itself for each $x$                    | 1 |
| <b>E1</b> | (d) so $C$ differs from every value of $f$                 | 1 |

**Why this question** This is the proof of Cantor's theorem run on a real example. The set  $C$  is built to disagree with  $f(x)$  at the one place where it matters, the element  $x$ , and that is why no function  $X \rightarrow \mathcal{P}(X)$  is ever surjective.

## 6 Cantor's set on $\mathbb{N}$

6 marks

- (a)  $f(0) = \mathbb{N}$  and  $f(3) = \{3, 4, 5, \dots\}$ .  
 (b) Every  $n$  lies in  $f(n)$ , because  $n \geq n$ . So  $C_f = \emptyset$ .  
 (c) **No**. Each value  $f(n)$  contains  $n$ , so it is never empty.  
 (d)  $n \notin \llbracket n \rrbracket$  for every  $n$ , so  $C_g = \mathbb{N}$ . It is **not a value of  $g$** , since each  $\llbracket n \rrbracket$  is missing  $n$ .

|           |                                    |   |
|-----------|------------------------------------|---|
| <b>B1</b> | (a) both sets                      | 1 |
| <b>M1</b> | (b) checking whether $n \in f(n)$  | 1 |
| <b>A1</b> | (b) $\emptyset$                    | 1 |
| <b>B1</b> | (c) no                             | 1 |
| <b>E1</b> | (c) each $f(n)$ contains $n$       | 1 |
| <b>B1</b> | (d) $\mathbb{N}$ , and not a value | 1 |

**Why this question** Cantor's set can be as small as  $\emptyset$  or as large as  $\mathbb{N}$ ; what matters is only that it is missed. Part (d) also shows  $g$  is not surjective, which agrees with  $\mathbb{N}$  being infinite.

## 7 Counting bijections and injections

5 marks

(a)  $4! = 24$ . (b)  $\mathcal{P}(\llbracket 1 \rrbracket) = \{\emptyset, \{0\}\}$  has 2 elements, so  $3 \times 2 = 6$ . (c)  $|\mathcal{P}(\llbracket 2 \rrbracket)| = 4$ , so **24**. (d) **0**: a function on  $\llbracket 3 \rrbracket$  takes at most 3 values, so it cannot hit all of  $\llbracket 4 \rrbracket$ .

|           |                                                          |   |
|-----------|----------------------------------------------------------|---|
| <b>B1</b> | (a) 24                                                   | 1 |
| <b>M1</b> | (b) finding $ \mathcal{P}(\llbracket 1 \rrbracket)  = 2$ | 1 |
| <b>A1</b> | (b) 6                                                    | 1 |
| <b>B1</b> | (c) 24                                                   | 1 |
| <b>B1</b> | (d) 0                                                    | 1 |

**Why this question** Only the sizes of the two sets matter for these counts, not what their elements are. That is the whole point of cardinality.

## 8 Cantor–Schröder–Bernstein in practice

6 marks

- (a) **Injective** ( $4n = 4m$  gives  $n = m$ ). **Not surjective**: it misses 2, since  $4n = 2$  has no solution in  $\mathbb{N}$ .
- (b) **Injective** (it does not change its input). **Not surjective**: it misses 1, which is odd.
- (c)  $n \mapsto 2n$ , with inverse  $m \mapsto m/2$ .

|           |                               |   |
|-----------|-------------------------------|---|
| <b>B1</b> | (a) injective, not surjective | 1 |
| <b>B1</b> | (a) a missed value, such as 2 | 1 |
| <b>B1</b> | (b) injective, not surjective | 1 |
| <b>B1</b> | (b) a missed value, such as 1 | 1 |
| <b>B1</b> | (c) bijection                 | 1 |
| <b>B1</b> | (c) inverse                   | 1 |

**Why this question** The Cantor–Schröder–Bernstein theorem promises a bijection whenever there are injections both ways, even when neither injection is a bijection itself. Here neither  $f$  nor  $g$  works, but a bijection still exists.

## 9 Finite or infinite?

6 marks

(a) **Finite**,  $2^{10} = 1024$ . (b) **Finite**, **32**: the elements are  $0, 1, \dots, 31$ , since  $31^2 = 961$  and  $32^2 = 1024$ . (c) **Infinite**. (d) **Finite**, **2**:  $\sqrt{2}$  and  $-\sqrt{2}$ . (e) **Finite**, **0**:  $\sqrt{2}$  is not rational, so the set is empty. (f) **Infinite**: it contains  $\{n\}$  for every  $n \in \mathbb{N}$ .

|           |                                                      |   |
|-----------|------------------------------------------------------|---|
| <b>B1</b> | each part correct, with the cardinality where finite | 6 |
|-----------|------------------------------------------------------|---|

**Common mistake** Part (e): the empty set is finite, with cardinality 0, because  $\llbracket 0 \rrbracket = \emptyset$ .

10 Why  $|\mathcal{P}(\llbracket n \rrbracket)| = 2^n$

7 marks

- (a)  $|F| = 2^3 = 8$  and  $|\mathcal{P}(\llbracket 3 \rrbracket)| = 8$ .
- (b)  $\Phi(f) = \{x \in \llbracket 3 \rrbracket \mid f(x) = 1\}$ : send each function to the set of points it sends to 1. The inverse sends a subset  $A$  to the function with  $f(x) = 1$  for  $x \in A$  and  $f(x) = 0$  otherwise.
- (c) The function  $(1, 0, 1)$  gives  $\{0, 2\}$ . The subset  $\{1\}$  gives the function  $f(0) = 0, f(1) = 1, f(2) = 0$ .
- (d) Each subset of  $\llbracket n \rrbracket$  is a yes or no choice for each of the  $n$  elements, which is a function  $\llbracket n \rrbracket \rightarrow \llbracket 2 \rrbracket$ , and there are  $2^n$  of those.

|    |                                               |   |
|----|-----------------------------------------------|---|
| B1 | (a) both 8                                    | 1 |
| M1 | (b) describing $\Phi$ by the points sent to 1 | 1 |
| A1 | (b) a clear rule for $\Phi$                   | 1 |
| B1 | (b) the inverse described                     | 1 |
| B1 | (c) $\{0, 2\}$                                | 1 |
| B1 | (c) the function $(0, 1, 0)$                  | 1 |
| E1 | (d)                                           | 1 |

**Why this question** This gives a clean reason why  $\llbracket n \rrbracket$  has  $2^n$  subsets. A subset is just a list of yes or no answers, one per element, which is the same as a binary string.

**11 Exam style: Cantor for a two-element set**

10 marks

- (a)  $\emptyset, \{a\}, \{b\}, \{a, b\}$ .
- (b)  $4^2 = 16$ .
- (c)  $a \in f(a)$ , so  $a \notin C$ .  $b \notin f(b) = \{a\}$ , so  $b \in C$ . So  $C = \{b\}$ . The values of  $f$  are  $\{a, b\}$  and  $\{a\}$ , **neither of which is  $\{b\}$** .
- (d) **None**. A function on a two-element set takes at most 2 values, but  $\mathcal{P}(X)$  has 4 elements. (This also follows from Cantor's theorem.)
- (e)  $f(a)$  can be any of the 4 subsets, and  $f(b)$  any of the other 3. So  $4 \times 3 = 12$ .

|           |                                                  |   |
|-----------|--------------------------------------------------|---|
| <b>B1</b> | (a)                                              | 1 |
| <b>B1</b> | (b) 16                                           | 1 |
| <b>M1</b> | (c) checking $x \in f(x)$ for $a$ and $b$        | 1 |
| <b>A1</b> | (c) $C = \{b\}$                                  | 1 |
| <b>B1</b> | (c) $C$ not a value                              | 1 |
| <b>B1</b> | (d) none                                         | 1 |
| <b>E1</b> | (d) size reason or Cantor's theorem              | 1 |
| <b>M1</b> | (e) 4 choices for $f(a)$                         | 1 |
| <b>M1</b> | (e) 3 choices for $f(b)$ , different from $f(a)$ | 1 |
| <b>A1</b> | (e) 12                                           | 1 |

**Why this question** This joins counting from Topic 2 to Cantor's theorem. For finite sets you can see Cantor's theorem by counting; the diagonal set  $C$  is what makes it work for infinite sets too.

**12 Exam style: a bijection onto the non-multiples of 3**

9 marks

- (a)  $n \mapsto n + 1$ , with inverse  $m \mapsto m - 1$ .
- (b)  $h(0) = 1, h(1) = 2, h(2) = 4, h(3) = 5, h(4) = 7, h(5) = 8$ .
- (c) Every element of  $T$  has the form  $3k + 1$  or  $3k + 2$  for exactly one  $k \in \mathbb{N}$ . So  $h^{-1}(3k + 1) = 2k$  and  $h^{-1}(3k + 2) = 2k + 1$ . Check:  $h(4) = 7 = 3 \cdot 2 + 1 \mapsto 4$  and  $h(5) = 8 = 3 \cdot 2 + 2 \mapsto 5$ .
- (d)  $10 = 3 \cdot 3 + 1$ , so  $n = 6$ .  $11 = 3 \cdot 3 + 2$ , so  $n = 7$ .

|           |                                                        |   |
|-----------|--------------------------------------------------------|---|
| <b>B1</b> | (a) bijection                                          | 1 |
| <b>B1</b> | (a) inverse                                            | 1 |
| <b>B1</b> | (b) all six values                                     | 1 |
| <b>M1</b> | (c) splitting $T$ into the forms $3k + 1$ and $3k + 2$ | 1 |
| <b>A1</b> | (c) $h^{-1}(3k + 1) = 2k$                              | 1 |
| <b>A1</b> | (c) $h^{-1}(3k + 2) = 2k + 1$                          | 1 |
| <b>B1</b> | (c) both checks                                        | 1 |
| <b>B1</b> | (d) $n = 6$                                            | 1 |
| <b>B1</b> | (d) $n = 7$                                            | 1 |

**Why this question** The values in (b) run through  $T$  in order:  $1, 2, 4, 5, 7, 8, \dots$ . Listing an infinite set in order, with nothing skipped and nothing repeated, is exactly what a bijection from  $\mathbb{N}$  does.

Total: 74 marks. Spotted a mistake? Report it at [unimixedtutor.com](https://unimixedtutor.com) and we will fix it.