

Topic 2 · Sheet A: Questions

Functions: injective, surjective, bijective, composition

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topic 1

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need functions, images and preimages, injections, surjections, bijections, composition and left and right inverses, plus anything from Topic 1.

Notation. A function $f: X \rightarrow Y$ has a domain X , a codomain Y and a rule; two functions are equal only if all three agree. We write $g \circ f$ for g composed with f , so $(g \circ f)(x) = g(f(x))$, and Id_X for the identity function on X . For a subset A of the domain, $f(A)$ is its image; for a subset B of the codomain, $f^{-1}(B)$ is its preimage. $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.

1

4 marks

Let $f: \llbracket 4 \rrbracket \rightarrow \llbracket 4 \rrbracket$ be given by $f(0) = 2, f(1) = 0, f(2) = 2$ and $f(3) = 1$.

- (a) What is the image of 3 under f ? [1]
- (b) List every preimage of 2 under f . [1]
- (c) Which elements of the codomain have no preimage? [1]
- (d) Is f injective? Is f surjective? [1]

2

4 marks

Answer each part, giving a one-line reason.

- (a) Let $f: \llbracket 3 \rrbracket \rightarrow \llbracket 3 \rrbracket$ be given by $f(x) = x$. Is $f = \text{Id}_{\llbracket 3 \rrbracket}$? [1]
- (b) Let $h: \llbracket 3 \rrbracket \rightarrow \llbracket 4 \rrbracket$ be given by $h(x) = x$. Is $h = \text{Id}_{\llbracket 3 \rrbracket}$? [1]
- (c) Does the rule $k(x) = x^2$ give a function $\llbracket 3 \rrbracket \rightarrow \llbracket 3 \rrbracket$? [1]
- (d) Does the rule $k(x) = x^2$ give a function $\llbracket 2 \rrbracket \rightarrow \llbracket 2 \rrbracket$? If so, is it equal to $\text{Id}_{\llbracket 2 \rrbracket}$? [1]

3

6 marks

Count the functions by counting the choices for each value.

- (a) List every function on $\llbracket 2 \rrbracket$, by giving the images of 0 and 1. [2]
- (b) How many functions are there from $\llbracket 3 \rrbracket$ to $\llbracket 2 \rrbracket$? [1]

- (c) How many functions are there from $\llbracket 2 \rrbracket$ to $\llbracket 3 \rrbracket$? [1]
- (d) How many functions are there on $\llbracket 4 \rrbracket$? [1]
- (e) How many functions are there from $\llbracket 1 \rrbracket$ to $\llbracket 5 \rrbracket$? [1]

4

6 marks

Decide whether each function is injective, surjective, both, or neither.

- (a) $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 1$ [1]
- (b) $g: \mathbb{N} \rightarrow \mathbb{N}, g(n) = 2n$ [1]
- (c) $h: \mathbb{N} \rightarrow \mathbb{N}$, where $h(0) = 0$ and $h(n) = n - 1$ for $n \geq 1$ [1]
- (d) $k: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}, k(x) = x^2$ [1]
- (e) $p: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, p(x) = x^2$ [1]
- (f) $q: \llbracket 4 \rrbracket \rightarrow \llbracket 4 \rrbracket, q(x) = 3 - x$ [1]

5

6 marks

Let $f, g: \llbracket 4 \rrbracket \rightarrow \llbracket 4 \rrbracket$ be given by the table below.

x	0	1	2	3
$f(x)$	1	2	3	0
$g(x)$	0	0	3	2

- (a) Give the table of $g \circ f$. [2]
- (b) Give the table of $f \circ g$. [2]
- (c) Is $g \circ f = f \circ g$? [1]
- (d) Is $f \circ f \circ f \circ f = \text{Id}_{\llbracket 4 \rrbracket}$? [1]

6

6 marks

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$, and let $g: \mathbb{N} \rightarrow \mathbb{N}$ be given by $g(n) = 2n$. Find each set.

- (a) $f(\{-2, -1, 0, 1, 2\})$ [1]
- (b) $f^{-1}(\{4\})$ [1]
- (c) $f^{-1}(\{-1\})$ [1]
- (d) $f^{-1}(\{0, 1\})$ [1]
- (e) $g^{-1}(\{3\})$ and $g^{-1}(\{0, 1, 2, 3, 4\})$ [2]

7

5 marks

- (a) How many bijections are there from $\llbracket 3 \rrbracket$ to $\llbracket 3 \rrbracket$? List each one as the list of images $f(0), f(1), f(2)$. [2]
- (b) How many injections are there from $\llbracket 2 \rrbracket$ to $\llbracket 4 \rrbracket$? Explain your count. [1]
- (c) How many surjections are there from $\llbracket 3 \rrbracket$ to $\llbracket 2 \rrbracket$? [2]

8

6 marks

Let $f: \mathbb{Z}_2 \rightarrow \mathbb{Z}_3$ be given by $f(0) = 0$ and $f(1) = 2$.

- (a) Is f injective? Is f surjective? [1]
- (b) Find a function $g: \mathbb{Z}_3 \rightarrow \mathbb{Z}_2$ with $g \circ f = \text{Id}_{\mathbb{Z}_2}$. [2]
- (c) How many such functions g are there? [1]
- (d) Does f have a right inverse? Explain using the link between right inverses and surjectivity. [2]

9

8 marks

Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 3x - 6$ and $g(x) = x^3$.

- (a) Find a formula for f^{-1} , and check that $f^{-1} \circ f = \text{Id}_{\mathbb{R}}$. [2]
- (b) Find $(g \circ f)(2)$ and $(f \circ g)(2)$. [1]
- (c) Find formulas for $(g \circ f)(x)$ and $(f \circ g)(x)$. [2]
- (d) The function $g \circ f$ is a bijection. Find a formula for $(g \circ f)^{-1}$. [3]

10

7 marks

Recall that a permutation of X is a bijection $X \rightarrow X$.

- (a) How many permutations of $\mathbb{Z}_4$ are there? [1]
- (b) How many permutations σ of $\mathbb{Z}_4$ have $\sigma(0) = 0$? [1]
- (c) How many permutations σ of $\mathbb{Z}_4$ have $\sigma(x) \neq x$ for every x ? List them systematically. [3]
- (d) How many permutations σ of $\mathbb{Z}_4$ satisfy $\sigma \circ \sigma = \text{Id}_{\mathbb{Z}_4}$? [2]

11

10 marks

Exam style. Let $f, g: \mathbb{N} \rightarrow \mathbb{N}$ be given by

$$f(n) = n + 2, \quad g(n) = \begin{cases} n - 2 & \text{if } n \geq 2, \\ 0 & \text{if } n < 2. \end{cases}$$

- (a) Decide whether each of f and g is injective, and whether each is surjective. Justify all four answers. [4]
- (b) Find $g \circ f$ and $f \circ g$. [2]
- (c) Which of f and g is a left inverse of the other? [1]
- (d) Find a left inverse h of f with $h \neq g$. [2]
- (e) How many left inverses does f have? [1]

Exam style. Let $X = \llbracket 3 \rrbracket$ and $Y = \llbracket 5 \rrbracket$.

- (a) How many functions are there from X to Y ? [1]
- (b) How many of them are injective? Explain your count. [2]
- (c) Are any of them surjective? Explain. [2]
- (d) How many functions $f: X \rightarrow Y$ satisfy $f(0) < f(1) < f(2)$? [3]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.