

## Topic 2 · Sheet A: Worked answers and mark scheme

# Functions: injective, surjective, bijective, composition

Suggested time

**About 75 minutes**

Total marks

**76**

Questions

**12**

Builds on

**Topic 1**

**How marks are awarded.** Each mark has a code, as on most UK exam papers.

**B** A mark for a correct statement or value on its own. It does not depend on any other mark.

**M** A method mark, for a correct approach, even if the final answer is wrong.

**A** An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

**E** An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

## 1 Images and preimages from a table

4 marks

(a)  $f(3) = 1$ . (b) **0 and 2**, since  $f(0) = f(2) = 2$ . (c) **3**: no  $x$  has  $f(x) = 3$ . (d) **Neither**. Not injective since  $f(0) = f(2)$ ; not surjective since 3 has no preimage.

|           |                                                    |   |
|-----------|----------------------------------------------------|---|
| <b>B1</b> | (a) 1                                              | 1 |
| <b>B1</b> | (b) both 0 and 2                                   | 1 |
| <b>B1</b> | (c) 3                                              | 1 |
| <b>B1</b> | (d) not injective and not surjective, both correct | 1 |

**Why this question** Reading a function from a table is the quickest way to build intuition. Injective means no value of the codomain is hit twice; surjective means none is missed. Part (b) shows a value hit twice, part (c) a value missed.

## 2 When are functions equal?

4 marks

(a) **Yes**. Same domain, same codomain, and  $f(x) = x$  for every  $x$ .

(b) **No**. The codomain of  $h$  is  $\llbracket 4 \rrbracket$ , not  $\llbracket 3 \rrbracket$ , so the functions differ even though the rule is the same.

(c) **No.**  $k(2) = 4$ , which is not in  $\llbracket 3 \rrbracket$ .

(d) **Yes, and yes.**  $0^2 = 0$  and  $1^2 = 1$  both lie in  $\llbracket 2 \rrbracket$ , and these are exactly the values of  $\text{Id}_{\llbracket 2 \rrbracket}$ .

|           |                                           |   |
|-----------|-------------------------------------------|---|
| <b>B1</b> | (a) yes, with reason                      | 1 |
| <b>B1</b> | (b) no, citing the codomain               | 1 |
| <b>B1</b> | (c) no, citing $k(2) = 4$                 | 1 |
| <b>B1</b> | (d) a function, and equal to the identity | 1 |

**Why this question** The squaring map  $x \mapsto x^2$  is the classic example: the rule alone does not determine a function. Part (b) is the one most people get wrong.

### 3 Counting functions

6 marks

(a) **Four functions**, written as  $(f(0), f(1))$ :  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 0)$ ,  $(1, 1)$ .

(b)  $2^3 = 8$ . Each of the three elements has two possible images.

(c)  $3^2 = 9$ .

(d)  $4^4 = 256$ .

(e) 5, one for each possible image of 0.

|           |                                                             |   |
|-----------|-------------------------------------------------------------|---|
| <b>B2</b> | (a) all four. <i>B1 for three correct, no wrong entries</i> | 2 |
| <b>B1</b> | (b) 8                                                       | 1 |
| <b>B1</b> | (c) 9                                                       | 1 |
| <b>B1</b> | (d) 256                                                     | 1 |
| <b>B1</b> | (e) 5                                                       | 1 |

**Common mistake** Mixing up (b) and (c). The number of functions from  $\llbracket m \rrbracket$  to  $\llbracket n \rrbracket$  is  $n^m$ : each of the  $m$  elements of the *domain* chooses one of  $n$  images.

### 4 Classifying functions

6 marks

(a) **Both (bijective)**. Each  $y$  has exactly one preimage,  $x = (y - 1)/2$ .

(b) **Injective only**.  $2n = 2m$  gives  $n = m$ , but 1 is never hit.

(c) **Surjective only**. Every  $m$  is  $h(m + 1)$ , but  $h(0) = h(1) = 0$ .

(d) **Surjective only**. Every  $y \geq 0$  is  $(\sqrt{y})^2$ , but  $k(-1) = k(1)$ .

(e) **Both**. Each  $y \geq 0$  has exactly one non-negative square root.

(f) **Both**. The table of  $q$  is 3, 2, 1, 0, which hits each value once.

|           |                   |   |
|-----------|-------------------|---|
| <b>B1</b> | each part correct | 6 |
|-----------|-------------------|---|

**Why this question** Parts (d) and (e) show the key idea: shrinking the domain can make a function injective, and shrinking the codomain can make it surjective.

## 5 Composition from tables

6 marks

| $x$              | 0 | 1 | 2 | 3 |
|------------------|---|---|---|---|
| $(g \circ f)(x)$ | 0 | 3 | 2 | 0 |
| $(f \circ g)(x)$ | 1 | 1 | 0 | 3 |

For example  $(g \circ f)(1) = g(f(1)) = g(2) = 3$ , and  $(f \circ g)(2) = f(g(2)) = f(3) = 0$ .

(c) **No**; for example they differ at 0.

(d) **Yes**.  $f$  sends each  $x$  to the next value round the cycle  $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$ , so four steps return every  $x$  to itself.

|           |                                              |   |
|-----------|----------------------------------------------|---|
| <b>M1</b> | (a) correct order: apply $f$ first, then $g$ | 1 |
| <b>A1</b> | (a) all four values                          | 1 |
| <b>M1</b> | (b) apply $g$ first, then $f$                | 1 |
| <b>A1</b> | (b) all four values                          | 1 |
| <b>B1</b> | (c) no, with a value where they differ       | 1 |
| <b>B1</b> | (d) yes                                      | 1 |

**Common mistake** Applying the functions in the wrong order. In  $g \circ f$ , the function written on the *right* acts first.

## 6 Images and preimages of sets

6 marks

(a)  $\{0, 1, 4\}$ . Repeated values are only listed once.

(b)  $\{-2, 2\}$ .

(c)  $\emptyset$ : no real square is negative.

(d)  $\{-1, 0, 1\}$ .

(e)  $g^{-1}(\{3\}) = \emptyset$  and  $g^{-1}(\{0, 1, 2, 3, 4\}) = \{0, 1, 2\}$ .

|           |                              |   |
|-----------|------------------------------|---|
| <b>B1</b> | (a)                          | 1 |
| <b>B1</b> | (b)                          | 1 |
| <b>B1</b> | (c) $\emptyset$              | 1 |
| <b>B1</b> | (d)                          | 1 |
| <b>B1</b> | (e) first set $\emptyset$    | 1 |
| <b>B1</b> | (e) second set $\{0, 1, 2\}$ | 1 |

**Why this question** Preimages exist for every function, even ones with no inverse. Parts (c) and (e) show that a preimage can be empty, which is exactly why the notation  $f^{-1}$  has to be read with care.

## 7 Counting special functions

5 marks

- (a) **6**:  $(0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 2, 0), (2, 0, 1), (2, 1, 0)$ .
- (b)  $4 \times 3 = 12$ .  $f(0)$  can be any of 4 values;  $f(1)$  must then avoid it, leaving 3.
- (c) **6**. There are  $2^3 = 8$  functions, and the only two that miss a value are the constant functions.

|           |                                                                 |   |
|-----------|-----------------------------------------------------------------|---|
| <b>B2</b> | (a) 6 with all listed. <i>B1 for 6 with an incomplete list</i>  | 2 |
| <b>B1</b> | (b) 12, with the $4 \times 3$ reasoning                         | 1 |
| <b>M1</b> | (c) counting all functions and removing the non-surjective ones | 1 |
| <b>A1</b> | (c) 6                                                           | 1 |

**Why this question** Part (c) introduces counting by complement: it is often easier to count what you do *not* want and subtract.

## 8 Left and right inverses

6 marks

- (a) **Injective, but not surjective**: 1 has no preimage.
- (b) We need  $g(f(0)) = 0$  and  $g(f(1)) = 1$ , that is  $g(0) = 0$  **and**  $g(2) = 1$ . The value  $g(1)$  can be either 0 or 1; for example  $g(0) = 0, g(1) = 0, g(2) = 1$ .
- (c) **2**, one for each choice of  $g(1)$ .
- (d) **No**. A function has a right inverse only if it is surjective, and it is not. Directly:  $f \circ h = \text{Id}_{[3]}$  would need  $f(h(1)) = 1$ , but 1 is not a value of  $f$ .

|           |                                                             |   |
|-----------|-------------------------------------------------------------|---|
| <b>B1</b> | (a) injective, not surjective                               | 1 |
| <b>M1</b> | (b) conditions $g(0) = 0, g(2) = 1$                         | 1 |
| <b>A1</b> | (b) a complete correct $g$                                  | 1 |
| <b>B1</b> | (c) 2                                                       | 1 |
| <b>B1</b> | (d) no                                                      | 1 |
| <b>E1</b> | (d) reason: $f$ is not surjective, or 1 is not in the image | 1 |

**Why this question** This makes the link between inverses and injectivity concrete. A left inverse only has to undo  $f$  on the values  $f$  actually takes, so the leftover value  $g(1)$  is a free choice. That is why left inverses are not unique.

## 9 Inverses and composition with formulas

8 marks

- (a) Solving  $y = 3x - 6$  gives  $f^{-1}(y) = \frac{y+6}{3}$ . Check:  $f^{-1}(f(x)) = \frac{(3x-6)+6}{3} = x$ .
- (b)  $(g \circ f)(2) = g(0) = 0$  and  $(f \circ g)(2) = f(8) = 18$ .
- (c)  $(g \circ f)(x) = (3x-6)^3$  and  $(f \circ g)(x) = 3x^3 - 6$ .
- (d) Solving  $y = (3x-6)^3$ : take cube roots, then undo  $f$ . So  $(g \circ f)^{-1}(y) = \frac{\sqrt[3]{y}+6}{3}$ .

|    |                                              |   |
|----|----------------------------------------------|---|
| B1 | (a) formula for $f^{-1}$                     | 1 |
| B1 | (a) correct check                            | 1 |
| B1 | (b) both values                              | 1 |
| B1 | (c) $(g \circ f)(x)$                         | 1 |
| B1 | (c) $(f \circ g)(x)$                         | 1 |
| M1 | (d) undoing $g$ first, by taking a cube root | 1 |
| M1 | (d) then applying $f^{-1}$ from (a)          | 1 |
| A1 | (d) correct formula                          | 1 |

**Why this question** Part (d) shows that  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ : to undo “first  $f$ , then  $g$ ”, you undo  $g$  first. Think of taking off shoes, then socks.

## 10 Counting permutations

7 marks

- (a)  $4 \times 3 \times 2 \times 1 = 24$ .
- (b) 6:  $\sigma$  is then a permutation of the remaining three elements.
- (c) 9. Listed by images  $(\sigma(0), \sigma(1), \sigma(2), \sigma(3))$ :  $(1, 0, 3, 2)$ ,  $(1, 2, 3, 0)$ ,  $(1, 3, 0, 2)$ ,  $(2, 0, 3, 1)$ ,  $(2, 3, 0, 1)$ ,  $(2, 3, 1, 0)$ ,  $(3, 0, 1, 2)$ ,  $(3, 2, 0, 1)$ ,  $(3, 2, 1, 0)$ .
- (d) 10. The identity; the 6 that swap one pair and fix the rest; and the 3 that swap two disjoint pairs.

|    |                                                            |   |
|----|------------------------------------------------------------|---|
| B1 | (a) 24                                                     | 1 |
| B1 | (b) 6                                                      | 1 |
| M1 | (c) a systematic list, e.g. ordered by $\sigma(0)$         | 1 |
| A1 | (c) all nine listed, nothing extra                         | 1 |
| A1 | (c) 9                                                      | 1 |
| M1 | (d) recognising $\sigma$ can only fix points or swap pairs | 1 |
| A1 | (d) 10                                                     | 1 |

**Why this question** Part (d) rewards thinking about the shape of a permutation rather than listing all 24. If  $\sigma(\sigma(x)) = x$ , then either  $\sigma(x) = x$  or  $x$  and  $\sigma(x)$  swap. This idea returns in the permutations topic later.

## 11 Exam style: one-sided inverses on $\mathbb{N}$

10 marks

- (a)  **$f$ : injective, not surjective** (0 and 1 are never hit).  **$g$ : surjective, not injective** (every  $m$  is  $g(m+2)$ , but  $g(0) = g(1) = g(2) = 0$ ).
- (b)  **$g \circ f = \text{Id}_{\mathbb{N}}$** , since  $g(n+2) = n$ .  **$(f \circ g)(n) = n$  for  $n \geq 2$ , and  $(f \circ g)(0) = (f \circ g)(1) = 2$** , so  $f \circ g \neq \text{Id}_{\mathbb{N}}$ .
- (c)  **$g$  is a left inverse of  $f$**  (equivalently,  $f$  is a right inverse of  $g$ ).  $f$  is not a left inverse of  $g$ .
- (d) Any  $h$  with  $h(n) = n - 2$  for  $n \geq 2$  works, whatever  $h(0)$  and  $h(1)$  are. For example  **$h(0) = h(1) = 5$  and  $h(n) = n - 2$  for  $n \geq 2$** .
- (e) **Infinitely many**:  $h(0)$  and  $h(1)$  can be any natural numbers.

|           |                                                 |   |
|-----------|-------------------------------------------------|---|
| <b>B1</b> | (a) $f$ injective, with reason                  | 1 |
| <b>B1</b> | (a) $f$ not surjective: 0 and 1 are missed      | 1 |
| <b>B1</b> | (a) $g$ surjective: $m = g(m+2)$                | 1 |
| <b>B1</b> | (a) $g$ not injective: $g(0) = g(1)$            | 1 |
| <b>B1</b> | (b) $g \circ f = \text{Id}_{\mathbb{N}}$        | 1 |
| <b>B1</b> | (b) $f \circ g$ correct, including at 0 and 1   | 1 |
| <b>B1</b> | (c)                                             | 1 |
| <b>M1</b> | (d) seeing that only $h(0)$ and $h(1)$ are free | 1 |
| <b>A1</b> | (d) a valid $h \neq g$                          | 1 |
| <b>B1</b> | (e) infinitely many                             | 1 |

**Common mistake** Assuming that  $g \circ f = \text{Id}$  forces  $f \circ g = \text{Id}$ . That only happens for bijections. Here one composite is the identity and the other is not.

**12 Exam style: counting functions between finite sets**

8 marks

- (a)  $5^3 = 125$ .
- (b)  $5 \times 4 \times 3 = 60$ .  $f(0)$  has 5 choices,  $f(1)$  must differ from it, and  $f(2)$  must differ from both.
- (c) **No**. A function on  $\llbracket 3 \rrbracket$  takes at most 3 different values, but there are 5 values in  $\llbracket 5 \rrbracket$  to hit.
- (d) **10**. Such a function is fixed by choosing which three values it takes, then listing them in increasing order. The possible sets of values are  $\{0, 1, 2\}$ ,  $\{0, 1, 3\}$ ,  $\{0, 1, 4\}$ ,  $\{0, 2, 3\}$ ,  $\{0, 2, 4\}$ ,  $\{0, 3, 4\}$ ,  $\{1, 2, 3\}$ ,  $\{1, 2, 4\}$ ,  $\{1, 3, 4\}$ ,  $\{2, 3, 4\}$ .

|           |                                                                    |   |
|-----------|--------------------------------------------------------------------|---|
| <b>B1</b> | (a) 125                                                            | 1 |
| <b>M1</b> | (b) the $5 \times 4 \times 3$ reasoning                            | 1 |
| <b>A1</b> | (b) 60                                                             | 1 |
| <b>B1</b> | (c) no                                                             | 1 |
| <b>E1</b> | (c) at most 3 values are hit                                       | 1 |
| <b>M1</b> | (d) seeing that the function is decided by its set of three values | 1 |
| <b>B1</b> | (d) a systematic list                                              | 1 |
| <b>A1</b> | (d) 10                                                             | 1 |

**Why this question** Part (d) is the key idea: an increasing function is the same thing as a choice of subset. Each 3-element subset of  $\llbracket 5 \rrbracket$  gives exactly one such function, and every such function comes from one subset.

Total: 76 marks. Spotted a mistake? Report it at [unimixedtutor.com](https://unimixedtutor.com) and we will fix it.