

Topic 1 · Sheet A: Questions

Sets, subsets and power sets

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Nothing earlier

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You only need sets, the empty set, subsets, power sets and specification.

Notation. In this module, $\mathbb{N} = \{0, 1, 2, \dots\}$ includes zero, $\llbracket n \rrbracket = \{0, 1, \dots, n-1\}$, and $X \subset Y$ means that X is a subset of Y (possibly equal to Y). You may use the fact that $\llbracket n \rrbracket$ has 2^n subsets. Where a and b appear, they are two different objects, and neither of them is a set.

1

4 marks

Decide whether each statement is true or false.

- (a) $3 \in \llbracket 3 \rrbracket$ [1]
- (b) $0 \in \llbracket 1 \rrbracket$ [1]
- (c) $\frac{1}{2} \in \mathbb{N}$ [1]
- (d) $\llbracket 0 \rrbracket = \emptyset$ [1]

2

4 marks

Write each set as a list of its elements.

- (a) $\{k \in \mathbb{N} \mid k < 5 \text{ and } k \text{ is odd}\}$ [1]
- (b) $\{k \in \llbracket 30 \rrbracket \mid k = m^2 \text{ for some } m \in \mathbb{N}\}$ [1]
- (c) $\{k \in \llbracket 10 \rrbracket \mid k \geq 10\}$ [1]
- (d) $\{k \in \mathbb{N} \mid k^2 < 10\}$ [1]

3

4 marks

For each pair, decide whether the two sets are equal, and give a one-line reason.

- (a) $\{0, 1, 2\}$ and $\{2, 1, 0, 1\}$ [1]
- (b) $\{0, \{1\}\}$ and $\{0, 1\}$ [1]
- (c) $\llbracket 4 \rrbracket$ and $\{k \in \mathbb{N} \mid k^2 < 10\}$ [1]
- (d) $\emptyset$ and $\{\emptyset\}$ [1]

4

7 marks

Suppose that $X = \{a, \{a\}, \{a, \{b\}\}\}$. Decide whether each of statements (a) to (f) is true or false, then answer (g).

- (a) $a \in X$ [1]
- (b) $\{a\} \in X$ [1]
- (c) $\{a\} \subset X$ [1]
- (d) $\{b\} \in X$ [1]
- (e) $\{\{a\}\} \subset X$ [1]
- (f) $\{a, \{b\}\} \subset X$ [1]
- (g) How many elements does X have? [1]

5

4 marks

Each statement below is false. Change exactly one symbol in each so that it becomes true. You may only replace $\in$ by $\subset$, or $\subset$ by $\in$.

- (a) $\{0, 1\} \in \mathbb{[3]}$ [1]
- (b) $0 \subset \mathbb{[3]}$ [1]
- (c) $\emptyset \in \mathbb{[3]}$ [1]
- (d) $\{1\} \in \mathcal{P}(\mathbb{[3]})$ and $\{1\} \in \mathbb{[3]}$, changing only the second symbol [1]

6

6 marks

Let $X = \mathbb{[3]}$.

- (a) List every element of $\mathcal{P}(X)$. [2]
- (b) How many elements of $\mathcal{P}(X)$ are singleton sets? [1]
- (c) How many elements of $\mathcal{P}(X)$ contain 0 as an element? [1]
- (d) Draw the Hasse diagram of $\mathcal{P}(X)$, as in the lectures: place the subsets in rows by size, and join A to B by an ascending arc when $A \subset B$ and B has exactly one more element than A . How many arcs does your diagram have? [2]

7

5 marks

Let $X = \mathbb{[5]}$.

- (a) How many elements does $\mathcal{P}(X)$ have? [1]
- (b) How many subsets of X are singleton sets? [1]
- (c) How many subsets of X contain both 0 and 1? Explain your count. [2]
- (d) How many subsets of X do not contain 4? [1]

8

6 marks

- (a) Write down $\mathcal{P}(\emptyset)$. [1]
- (b) Write down $\mathcal{P}(\{\emptyset\})$. [1]
- (c) List every element of $\mathcal{P}(\{\emptyset, \{\emptyset\}\})$. [2]
- (d) How many elements does $\mathcal{P}(\mathcal{P}(\llbracket 2 \rrbracket))$ have? [1]
- (e) How many elements does $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ have? [1]

9

7 marks

Let

$$Y = \{k \in \llbracket 12 \rrbracket \mid k \text{ is a multiple of } 3\}, \quad Z = \{k \in \llbracket 12 \rrbracket \mid k + 3 \in \llbracket 12 \rrbracket \text{ and } k \text{ is a multiple of } 3\}.$$

- (a) List the elements of Y . [1]
- (b) List the elements of Z . [1]
- (c) Is $Z \subset Y$? Is $Y \subset Z$? Justify each answer. [2]
- (d) How many sets W satisfy both $Z \subset W$ and $W \subset Y$? List them and explain. [3]

10

7 marks

Let $X = \llbracket 4 \rrbracket$.

- (a) How many elements does $\mathcal{P}(X)$ have? [1]
- (b) How many subsets S of X satisfy $\{0\} \subset S$? [1]
- (c) List every subset S of X with $\{0, 1\} \subset S$. [2]
- (d) How many elements of $\mathcal{P}(X)$ are subsets of $\{0, 1\}$? [1]
- (e) How many elements does $\{S \in \mathcal{P}(X) \mid S \text{ has exactly two elements}\}$ have? List them. [2]

11

11 marks

Exam style. Suppose that $X = \{\emptyset, \{\emptyset\}, \{a, b\}\}$.

- (a) How many elements does X have? [1]
- (b) Decide whether each statement is true or false, giving a brief reason.
 - (i) $\emptyset \in X$ [1]
 - (ii) $\emptyset \subset X$ [1]
 - (iii) $\{\emptyset\} \subset X$ [1]
 - (iv) $\{a, b\} \subset X$ [1]
 - (v) $\{\{a, b\}\} \subset X$ [1]
- (c) How many elements does $\mathcal{P}(X)$ have? [1]
- (d) List the elements of $\mathcal{P}(X)$ that have $\emptyset$ as an element, and explain why there must be exactly that many. [4]

Exam style. Suppose that X is a set with exactly n elements, where $n \in \mathbb{N}$.

- (a) In terms of n , how many elements does $\mathcal{P}(\mathcal{P}(X))$ have? [1]
- (b) Find the smallest n for which $\mathcal{P}(\mathcal{P}(X))$ has more than 1000 elements, showing the values that justify your answer. [3]
- (c) Suppose that Y is a set and $\mathcal{P}(Y)$ has exactly 64 elements. How many elements does Y have, and how many elements of $\mathcal{P}(Y)$ are singleton sets? [2]
- (d) Is there a set W for which $\mathcal{P}(W)$ has exactly 12 elements? Explain. [2]
- (e) How many subsets of $\llbracket 6 \rrbracket$ contain 0 but do not contain 5? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.