

Topic 1 · Sheet A: Worked answers and mark scheme

Sets, subsets and power sets

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Nothing earlier

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Membership

4 marks

(a) **False.** $\llbracket 3 \rrbracket = \{0, 1, 2\}$ stops at 2. (b) **True.** $\llbracket 1 \rrbracket = \{0\}$. (c) **False.** $\mathbb{N}$ contains only whole numbers. (d) **True.** $\llbracket 0 \rrbracket = \{k \in \mathbb{N} \mid k < 0\}$ has no elements.

B1	(a) False	1
B1	(b) True	1
B1	(c) False	1
B1	(d) True	1

Why this question A warm-up on reading $\llbracket n \rrbracket$ correctly. The most common slip all term is treating $\llbracket n \rrbracket$ as $\{1, \dots, n\}$; it is $\{0, \dots, n-1\}$, so it has exactly n elements but never contains n itself.

2 Specification

4 marks

(a) $\{1, 3\}$ (b) $\{0, 1, 4, 9, 16, 25\}$ (c) $\emptyset$ (d) $\{0, 1, 2, 3\}$, since $3^2 = 9 < 10$ but $4^2 = 16$.

B1	(a) $\{1, 3\}$	1
B1	(b) all six squares, including 0, and nothing else	1
B1	(c) $\emptyset$; also accept $\{ \}$	1
B1	(d) $\{0, 1, 2, 3\}$	1

Common mistake Leaving out 0. In these notes $0 \in \mathbb{N}$, and $0 = 0^2$, so 0 belongs in (b) and (d). In (c), writing $\{\emptyset\}$ instead of $\emptyset$ scores 0: that set has one element.

3 Equality of sets

4 marks

- (a) **Equal.** By the axiom of extension only the elements matter, not their order or repeats; both sets are $\{0, 1, 2\}$.
- (b) **Not equal.** $1 \in \{0, 1\}$, but $1 \notin \{0, \{1\}\}$: its elements are 0 and the set $\{1\}$.
- (c) **Equal.** Both are $\{0, 1, 2, 3\}$ (see question 2(d)).
- (d) **Not equal.** $\emptyset$ has no elements, while $\{\emptyset\}$ has exactly one element, namely $\emptyset$.

B1	(a) Equal, with a reason mentioning order or repetition	1
B1	(b) Not equal, naming an element in one set but not the other	1
B1	(c) Equal, with both sets listed or compared	1
B1	(d) Not equal, noting $\{\emptyset\}$ has an element	1

Why this question This is the axiom of extension in action: to compare two sets, compare their elements and nothing else. Part (d) is the classic trap. A box with an empty box inside it is not an empty box.

4 Elements versus subsets

7 marks

The elements of X are exactly a , $\{a\}$ and $\{a, \{b\}\}$.

- (a) **True.** a is listed.
- (b) **True.** $\{a\}$ is listed.
- (c) **True.** Its only element a is in X .
- (d) **False.** $\{b\}$ is not one of the three elements; it is only an element of an element.
- (e) **True.** Its only element $\{a\}$ is in X .
- (f) **False.** $a \in X$, but $\{b\} \notin X$ by part (d).
- (g) **3.**

B1	each of (a) to (f) correct	6
B1	(g) 3	1

Why this question The key skill in this topic is telling $\in$ from $\subset$. To check $A \in X$, look for A in the list of elements of X . To check $A \subset X$, take each element of A in turn and look for it in that list.

5 Fix the symbol

4 marks

(a) $\{0,1\} \subset \mathbb{[3]}$ (b) $0 \in \mathbb{[3]}$ (c) $\emptyset \subset \mathbb{[3]}$ (d) $\{1\} \in \mathcal{P}(\mathbb{[3]})$ **and** $\{1\} \subset \mathbb{[3]}$

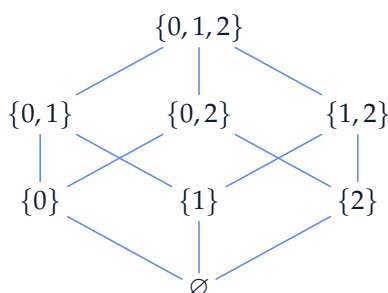
B1	each correct replacement	4
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Why this question Part (d) shows the link that defines the power set: $A \in \mathcal{P}(X)$ exactly when $A \subset X$. The same set $\{1\}$ is an *element* of $\mathcal{P}(\mathbb{[3]})$ and a *subset* of $\mathbb{[3]}$.

6 The power set of $\mathbb{[3]}$

6 marks

- (a) $\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}$, eight in all.
(b) **3:** $\{0\}, \{1\}$ and $\{2\}$.
(c) **4:** $\{0\}, \{0,1\}, \{0,2\}$ and $\{0,1,2\}$.
(d) The diagram below, which has **12 arcs**.



B2	(a) all eight subsets, no repeats. <i>B1 for 6 or 7 correct with no wrong entries</i>	2
B1	(b) 3	1
B1	(c) 4	1
B1	(d) diagram with four rows by size and the correct arcs	1
B1	(d) 12 arcs	1

Common mistake Forgetting $\emptyset$ or the whole set $\{0,1,2\}$. Both are subsets of $\mathbb{[3]}$, so both belong in $\mathcal{P}(\mathbb{[3]})$.

7 Counting subsets of $\llbracket 5 \rrbracket$

5 marks

- (a) **32**, since $\llbracket 5 \rrbracket$ has 2^5 subsets.
- (b) **5**, one for each element.
- (c) **8**. Any such subset is $\{0, 1\}$ together with some subset of $\{2, 3, 4\}$, and each of those $2^3 = 8$ choices gives a different subset.
- (d) **16**. These are exactly the subsets of $\llbracket 4 \rrbracket$, and there are 2^4 of them.

B1	(a) 32	1
B1	(b) 5	1
M1	(c) seeing that 0 and 1 are fixed and only 2, 3, 4 are free to choose	1
A1	(c) 8	1
B1	(d) 16	1

Why this question This introduces the main counting trick for the topic: fix the elements that must be in (or out), then count the free choices. Each free element doubles the count. Questions 10 and 12 build on it.

8 Power sets of small sets

6 marks

- (a) $\mathcal{P}(\emptyset) = \{\emptyset\}$: the empty set has exactly one subset, itself.
- (b) $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
- (c) $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$.
- (d) **16**. $\mathcal{P}(\llbracket 2 \rrbracket)$ has $2^2 = 4$ elements, so its power set has $2^4 = 16$.
- (e) **4**. The sizes go 1, then $2^1 = 2$, then $2^2 = 4$.

B1	(a) $\{\emptyset\}$; answer $\emptyset$ scores 0	1
B1	(b) both elements	1
B2	(c) all four. <i>B1 for three correct with no wrong entries</i>	2
B1	(d) 16	1
B1	(e) 4	1

Common mistake Writing $\mathcal{P}(\emptyset) = \emptyset$. The power set always contains the empty set, so it is never empty.

9 Sets between two sets

7 marks

- (a) $Y = \{0, 3, 6, 9\}$.
- (b) $Z = \{0, 3, 6\}$. We need $k + 3 \leq 11$, which rules out 9.
- (c) $Z \subset Y$: **yes**, since 0, 3 and 6 are all in Y . $Y \subset Z$: **no**, since $9 \in Y$ but $9 \notin Z$.
- (d) **Exactly 2**: $W = Z$ and $W = Y$. Every element of Z must be in W , and W can only add elements of Y . The only element of Y not already in Z is 9, so W either leaves it out ($W = Z$) or includes it ($W = Y$).

B1	(a) Y correct	1
B1	(b) Z correct	1
B1	(c) $Z \subset Y$, with reason	1
B1	(c) $Y \not\subset Z$, naming 9	1
M1	(d) noting W must contain every element of Z	1
M1	(d) noting W can only add elements of Y not in Z , i.e. 9	1
A1	(d) 2, with $W = Z$ and $W = Y$ listed	1

Why this question Specification can quietly remove elements, as $k + 3 \in \llbracket 12 \rrbracket$ does here. Part (d) is the counting trick from question 7 in disguise: one free element gives 2^1 sets.

10 Counting with conditions

7 marks

- (a) **16**.
- (b) **8**: $\{0\} \subset S$ just means $0 \in S$, leaving 1, 2, 3 free.
- (c) $\{0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 2, 3\}$.
- (d) **4**: $\emptyset$, $\{0\}$, $\{1\}$, $\{0, 1\}$.
- (e) **6**: $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$.

B1	(a) 16	1
B1	(b) 8	1
B2	(c) all four. <i>B1 for two or three correct with no wrong entries</i>	2
B1	(d) 4	1
B1	(e) 6	1
B1	(e) a correct list of the six	1

Why this question Part (b) is worth noticing: $\{x\} \subset S$ says the same as $x \in S$. Moving between the two forms is often what unlocks a question.

11 Exam style: a set of sets

11 marks

The elements of X are $\emptyset$, $\{\emptyset\}$ and $\{a, b\}$.

- (a) **3**.
- (b) (i) **True**. $\emptyset$ is listed as an element.
(ii) **True**. $\emptyset$ is a subset of every set.
(iii) **True**. Its only element, $\emptyset$, is in X .
(iv) **False**. a is not an element of X ; only the set $\{a, b\}$ is.
(v) **True**. Its only element, $\{a, b\}$, is in X .
- (c) **8**, since X has 3 elements.
- (d) $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{a, b\}\}$, $\{\emptyset, \{\emptyset\}, \{a, b\}\}$. These are the subsets of X that contain $\emptyset$, which leaves the other two elements free: $2^2 = 4$.

B1	(a) 3	1
B1	(b) each of (i) to (v) correct with a valid reason	5
B1	(c) 8	1
M1	(d) recognising these are the subsets of X containing $\emptyset$	1
M1	(d) two elements left free, so $2^2 = 4$	1
A2	(d) all four listed, nothing extra. <i>A1 for three correct with no wrong entries</i>	2

Common mistake Statements (i) and (ii) are both true, for different reasons. Many answers say only one of them can hold. Here $\emptyset$ happens to be an element of X , *and* it is a subset of X , as it is of every set.

12 Exam style: sizes of power sets

9 marks

- (a) 2^{2^n} . $\mathcal{P}(X)$ has 2^n elements, so its power set has $2^{(2^n)}$.
- (b) $n = 4$. The values for $n = 0, 1, 2, 3, 4$ are 2, 4, 16, 256, 65536. The first above 1000 is at $n = 4$.
- (c) Y has 6 elements, since $2^6 = 64$. 6 elements of $\mathcal{P}(Y)$ are singletons, one for each element of Y .
- (d) **No**. A set with n elements has a power set with 2^n elements, and 12 is not a power of 2: $2^3 = 8 < 12 < 16 = 2^4$.
- (e) **16**. Fix 0 in and 5 out; the four elements 1, 2, 3, 4 are free, giving 2^4 .

B1	(a) 2^{2^n}	1
M1	(b) computing 2^{2^n} for $n = 0, 1, 2, 3, 4$	1
M1	(b) comparing with 1000: $256 < 1000 < 65536$	1
A1	(b) $n = 4$	1
B1	(c) 6 elements	1
B1	(c) 6 singletons	1
B1	(d) No	1
E1	(d) reason: power set sizes are powers of 2, and 12 is not one	1
B1	(e) 16	1

Why this question This brings the whole sheet together: the 2^n rule, power sets of power sets, and counting with fixed elements. It is the kind of question that closes a test, with a quick start and a part (d) that asks you to say why, not just what.

Total: 74 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.