

Topic 15 · Sheet A: Questions

Trees and spanning trees

Suggested time

About 75 minutes

Total marks

73

Questions

12

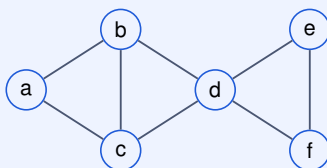
Builds on

Topics 1 to 14

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need trees and forests, pendant nodes (degree 1), the facts that a tree on n nodes has $n - 1$ edges and that a node of degree k forces at least k pendant nodes, spanning trees, unique paths in trees, cutpoints and bridges.

Notation. A *cutpoint* is a node whose removal disconnects the graph; a *bridge* is an edge whose removal does.

The graph G .



1

4 marks

Is each graph a tree?

- (a) P_5 [1]
- (b) C_4 [1]
- (c) The star with one centre joined to 4 other nodes [1]
- (d) A graph with 5 nodes and 4 edges that contains a triangle [1]

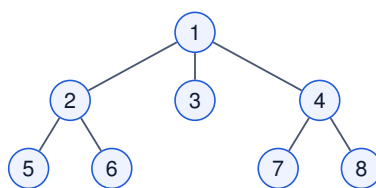
2

4 marks

- (a) How many edges does a tree with 10 nodes have? [1]
- (b) How many edges does a forest with 10 nodes and 3 connected components have? [1]
- (c) A connected graph has 10 nodes and 12 edges. How many edges must be removed to leave a spanning tree? [1]
- (d) A tree has a node of degree 5. What is the smallest possible number of pendant nodes? [1]

3

6 marks



This is the tree T .

- (a) List its pendant nodes. [2]
- (b) Check that it has $n - 1$ edges. [1]
- (c) What is left when the pendant node 3 and its edge are removed? [1]
- (d) Every tree is bipartite. Give the two parts for T . [2]

4

6 marks

A tree has two nodes of degree 3, one node of degree 4, and every other node is pendant.

- (a) Use $\sum_v \deg(v) = 2(n - 1)$ to find the number of pendant nodes. [3]
- (b) Draw such a tree. [2]
- (c) Check your answer to (a) against the result about a node of degree k . [1]

5

6 marks

These parts are about G .

- (a) How many edges must be removed from G to leave a spanning tree? [1]
- (b) Find a spanning tree by repeatedly removing an edge that lies on a cycle. Show each step. [3]
- (c) Is there an edge of G that belongs to every spanning tree? Explain. [2]

6

6 marks

These parts are about the tree T from question 3.

- (a) Write down the path from 3 to 7. Why is it the only one? [1]
- (b) The edge $\{5, 7\}$ is added. Give two different paths from 5 to 7, and the cycle they form. [2]
- (c) Without looking at cycles, explain why the new graph is not a tree. [1]
- (d) Which single edges could you remove from the new graph to get a tree again? [2]

7

6 marks

Count the spanning trees of each graph.

- (a) C_n [2]
- (b) K_4 , by counting sets of 3 edges and removing those that form a triangle [2]

- (c) The diamond: nodes 1, 2, 3, 4 and edges 12, 13, 23, 24, 34 [2]

8

6 marks

The graph H has nodes $a, \dots, h$. It consists of the triangle a, b, c , the edge cd , the triangle d, e, f , and the edges fg and gh .

- (a) List the cutpoints of H . [2]
- (b) List the nodes that are not cutpoints. [1]
- (c) Find a spanning tree of H and list its pendant nodes. [2]
- (d) Check that none of those pendant nodes is a cutpoint of H . [1]

9

6 marks

- (a) A forest has n nodes and k connected components. Show that it has $n - k$ edges. [3]
- (b) A graph has 12 nodes, 9 edges and no cycles. How many components does it have? [1]
- (c) Explain why a graph with 12 nodes and 10 edges cannot be connected. [2]

10

7 marks

The complete bipartite graph with parts $\{x, y\}$ and $\{1, 2, 3\}$ has 6 edges.

- (a) Find a cycle in it. [1]
- (b) Remove an edge of that cycle, find a cycle in what is left, and remove an edge of it. Explain why you now have a spanning tree. [3]
- (c) List the pendant nodes of your spanning tree. [2]
- (d) Check that the degrees of your tree add up to $2(n - 1)$. [1]

11

8 marks

Exam style. A tree has exactly $k \geq 1$ nodes of degree 3, and every other node has degree 1.

- (a) Show that it has $k + 2$ pendant nodes. [3]
- (b) How many nodes does it have in total? [1]
- (c) Draw such a tree with $k = 3$. [2]
- (d) Give the two parts of a bipartition of your tree. [2]

12

8 marks

Exam style. Six offices $A, \dots, F$ can be linked by the cables $AB, AC, BC, BD, CE, DE, DF, EF$. Every office must be able to reach every other, using as few cables as possible.

- (a) How many cables are needed? [1]
- (b) Choose a set of cables that works. [2]
- (c) In your network, write down the route from A to F . [1]

- (d) Explain why any 5 cables that connect all six offices contain no cycle. [2]
- (e) If one of your chosen cables fails, how many separate groups of offices are there? Explain. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.