

Topic 15 · Sheet A: Worked answers and mark scheme

Trees and spanning trees

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 14

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Tree or not

4 marks

(a) **Yes**. (b) **No**: it is a cycle. (c) **Yes**. (d) **No**: the triangle uses 3 edges on 3 nodes, leaving 1 edge for the other 2 nodes, so it cannot be connected (and it has a cycle).

B1 each part correct

4

Common mistake In (d), concluding “tree” because $4 = 5 - 1$. Having $n - 1$ edges only gives a tree when the graph is also connected, or also has no cycles.

2 Counting edges

4 marks

(a) **9**. (b) $10 - 3 = 7$. (c) $12 - 9 = 3$. (d) **5**.

B1 each part correct

4

Why this question A spanning tree of a connected graph on n nodes always has $n - 1$ edges, so the number of edges to remove is fixed, however you choose them.

3 Reading a tree

6 marks

- (a) 3, 5, 6, 7, 8.
- (b) 8 nodes and 7 edges.
- (c) A tree on the 7 nodes 1, 2, 4, 5, 6, 7, 8, with 6 edges.
- (d) {1, 5, 6, 7, 8} and {2, 3, 4}: nodes at even and odd distance from 1.

B1	(a) all five	1
B1	(a) no extras	1
B1	(b)	1
B1	(c)	1
B1	(d)	1
E1	(d) every edge checked	1

Why this question Removing a pendant node always leaves a smaller tree. That is why induction on the number of nodes works so well for trees.

4 How many pendant nodes?

6 marks

- (a) With ℓ pendant nodes, $n = \ell + 3$ and the degree sum is $3 + 3 + 4 + \ell$. So $10 + \ell = 2(\ell + 2)$, giving $\ell = 6$.
- (b) For example: the node of degree 4 joined to the two nodes of degree 3 and to two pendant nodes, with each node of degree 3 also joined to two pendant nodes. That uses 6 pendant nodes and 8 edges on 9 nodes.
- (c) The node of degree 4 forces at least 4 pendant nodes, and $6 \geq 4$.

M1	(a) n in terms of ℓ	1
M1	(a) degree sum equation	1
A1	(a)	1
B1	(b) correct degrees	1
B1	(b) is a tree	1
B1	(c)	1

Common mistake Using $\sum \deg = 2|E|$ with $|E|$ unknown. In a tree you also know $|E| = n - 1$, which is what makes the equation solvable.

5 A spanning tree of G

6 marks

- (a) $8 - (6 - 1) = 3$.
- (b) For example: remove bc (on the triangle a, b, c); remove cd (on the cycle a, b, d, c); remove ef (on the triangle d, e, f). Left: ab, ac, bd, de, df , which is connected with 5 edges on 6 nodes.
- (c) **No**. Every edge of G lies on a cycle, so for any edge there is a spanning tree that avoids it.

B1	(a)	1
M1	(b) each removed edge on a cycle	1
A1	(b) three valid removals	1
A1	(b) result is a spanning tree	1
B1	(c)	1
E1	(c)	1

Why this question Removing an edge that lies on a cycle never disconnects the graph: you can go round the rest of the cycle instead.

6 Unique paths

6 marks

- (a) $3, 1, 4, 7$. In a tree there is exactly one path between any two nodes.
- (b) $5, 7$ and $5, 2, 1, 4, 7$, which together form the cycle $5, 2, 1, 4, 7, 5$.
- (c) It has 8 nodes and 8 edges, but a tree on 8 nodes has exactly 7.
- (d) **Any edge of the cycle**: 25, 12, 14, 47 or 57.

B1	(a)	1
B1	(b) two paths	1
B1	(b) cycle	1
E1	(c)	1
B1	(d)	1
B1	(d) all five	1

Why this question Adding one edge to a tree creates exactly one cycle, and removing any edge of that cycle gives a tree again.

7 Counting spanning trees

6 marks

- (a) Removing any one of the n edges leaves a path: n .
- (b) $\binom{6}{3} = 20$ sets of 3 edges. The 4 triangles are not trees; every other set of 3 edges on 4 nodes is. So **16**.

(c) $\binom{5}{3} = 10$ sets of 3 edges, minus the 2 triangles 1,2,3 and 2,3,4: **8**.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake In (b), also removing the 4-cycles. A 4-cycle uses 4 edges, so it never appears among the sets of 3.

8 Cutpoints

6 marks

(a) *c, d, f and g*.

(b) *a, b, e and h*.

(c) For example remove *bc* and *ef*: the tree *ab, ac, cd, de, df, fg, gh*, with pendant nodes *b, e, h*.

(d) *b, e and h* are all in the list from (b). **None is a cutpoint.**

B1	(a)	1
B1	(a) no extras	1
B1	(b)	1
B1	(c) tree	1
B1	(c) pendant nodes	1
B1	(d)	1

Why this question Removing a pendant node of a spanning tree leaves the rest of the tree connected, so that node is never a cutpoint. This is why every connected graph has a node that is not a cutpoint.

9 Forests

6 marks

- (a) Each component is a tree. If the components have $n_1, \dots, n_k$ nodes, they have $n_1 - 1, \dots, n_k - 1$ edges, so the total is $(n_1 + \dots + n_k) - k = n - k$.
- (b) $12 - k = 9$, so **3**.
- (c) A connected graph on 12 nodes has at least **11 edges**, since it contains a spanning tree.

M1	(a) each component is a tree	1
M1	(a) summing	1
A1	(a)	1
B1	(b)	1
E2	(c)	2

Why this question “At least $n - 1$ edges” is the simplest test for connectedness: too few edges and the graph cannot be connected, whatever they are.

10 Pruning to a spanning tree

7 marks

- (a) For example $x, 1, y, 2, x$.
- (b) Remove $y2$. A cycle left is $x, 1, y, 3, x$; remove $y3$. Left: $x1, x2, x3, y1$: 4 edges on 5 nodes and still connected, so **a spanning tree**.
- (c) **2, 3 and y** .
- (d) Degrees x : 3, 1 : 2, $2, 3, y$: 1: sum $8 = 2 \cdot 4$.

B1	(a)	1
M1	(b) first removal	1
M1	(b) second removal	1
E1	(b) connected with $n - 1$ edges	1
B1	(c)	1
B1	(c)	1
B1	(d)	1

Why this question This is the proof that every connected graph has a spanning tree, used as an algorithm: keep removing cycle edges until no cycle is left.

11 Exam style: trees with degrees 1 and 3

8 marks

- (a) With ℓ pendant nodes, $n = k + \ell$ and the degree sum is $3k + \ell = 2(k + \ell - 1)$. Solving, $\ell = k + 2$.

- (b) $2k + 2$.
- (c) For example: a path u, v, w of three degree-3 nodes, with u and w each joined to two pendant nodes and v to one. That is 5 pendant nodes and 8 nodes in all.
- (d) In that tree: $\{u, w\}$ **together with v 's pendant node**, and $\{v\}$ **together with the four pendant nodes of u and w** .

M1	(a) $n = k + \ell$	1
M1	(a) degree sum $= 2(n - 1)$	1
A1	(a)	1
B1	(b)	1
B1	(c) correct degrees	1
B1	(c) is a tree	1
B1	(d)	1
E1	(d) checked	1

Common mistake In (d), putting all pendant nodes on one side. A pendant node goes on the opposite side to its only neighbour.

12 Exam style: a minimal network

8 marks

- (a) **5**, the number of edges of a spanning tree on 6 nodes.
- (b) For example AB, AC, BD, CE, DF .
- (c) A, B, D, F .
- (d) A connected graph on n nodes with $n - 1$ edges is a tree, so **it has no cycles**.
- (e) **2**. A tree has a unique path between any two nodes, so removing an edge leaves its two ends with no path between them, splitting the offices into two groups.

B1	(a)	1
B1	(b) five cables	1
B1	(b) connects everything	1
B1	(c)	1
E2	(d)	2
B1	(e)	1
E1	(e)	1

Why this question A spanning tree is the cheapest way to connect everything, but it has no backup: every link is a single point of failure.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.