

Topic 14 · Sheet A: Worked answers and mark scheme

Euler circuits and Euler walks

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 13

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Quick decisions

4 marks

- (a) All degrees 2: **circuit**. (b) Two ends of degree 1: **walk**. (c) Degrees 4, 4 and 2, 2, 2, 2: **circuit**.
 (d) Four odd degrees: **neither**.

B1 each part correct

4

Why this question Part (d) is the Seven Bridges of Königsberg. Counting odd degrees settles the question without trying any walks.

2 An Euler walk in G

4 marks

- (a) b **and** c (degree 3). (b) **No**. (c) **Yes, from b to c** (or the reverse). (d) For example $b, a, c, b, d, e, f, d, c$.

B1 each part correct

4

Common mistake Starting the walk at a . An Euler walk must start and end at the two odd nodes; starting elsewhere always gets stuck.

3 The house

6 marks

- (a) 1: 3, 2: 3, 3: 2, 4: 2, 5: 2.
- (b) Exactly two odd nodes, so **an Euler walk from 1 to 2, but no circuit.**
- (c) For example 1, 5, 2, 1, 4, 3, 2.

B1	(a)	1
B1	(a)	1
B1	(b)	1
E1	(b)	1
M1	(c) starting at an odd node	1
A1	(c)	1

Why this question This is the classic “draw the house without lifting your pen” puzzle. The degrees tell you where to start.

4 Splicing closed walks

6 marks

- (a) 34, 45 **and** 53.
- (b) 3, 4, 5, 3.
- (c) Insert it where the first walk reaches 3: 1, 2, 3, 4, 5, 3, 1.
- (d) It has 6 steps, and the bowtie has **6 edges.**

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) inserting at node 3	1
A1	(c)	1
B1	(d)	1

Why this question This is exactly the proof that even degrees give an Euler circuit: find a closed walk, then fit in circuits of what is left at places where they meet it.

5 Adding an edge

6 marks

- (a) b and c now have degree 4, so **every degree is even**, and G' is connected.
- (b) For example $b, a, c, b, d, e, f, d, c, b$, ending with the new edge $c \rightarrow b$.
- (c) $b, a, c, b, d, e, f, d, c$: an Euler walk of G from b to c .

B1	(a) even degrees	1
B1	(a) connected	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Adding an edge between the two odd nodes, finding a circuit, then removing the edge is how the Euler walk result is proved.

6 A grid

6 marks

- (a) q and t (degree 3). So there is **an Euler walk from q to t** , and no circuit.
- (b) For example q, p, s, t, q, r, u, t .
- (c) **A second edge between q and t** : then every degree is even.

B1	(a) odd nodes	1
B1	(a) conclusion	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
E1	(c)	1

Common mistake In (c), adding an edge from q to some other node. That makes the other node odd. The new edge must join the two odd nodes.

7 Cubes and bipartite graphs

6 marks

- (a) All 8 nodes have odd degree, so **neither**.
- (b) Six nodes of degree 3: **neither**.
- (c) The two nodes on the small side have degree 3; the other three have degree 2. Exactly two odd, so **an Euler walk between the two nodes on the side of size 2**.
- (d) All degrees 4: **an Euler circuit**.

B1	(a)	1
E1	(a)	1
B1	(b)	1
B1	(c)	1
E1	(c)	1
B1	(d)	1

Why this question In a complete bipartite graph, each node's degree is the size of the *other* side, so only the side sizes matter.

8 Drawing the envelope

6 marks

- (a) 1: 4, 2: 4, 3: 3, 4: 3, 5: 2.
- (b) The odd nodes are 3 and 4, so **start at one of them and finish at the other**.
- (c) For example 3, 4, 1, 2, 5, 1, 3, 2, 4.

B1	(a)	1
B1	(a)	1
B1	(b)	1
E1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake Counting the diagonals' crossing point as a node. Two lines crossing in a drawing do not meet at a node unless one is drawn there.

9 Bridges

6 marks

- (a) Nodes A, B, N, S ; one edge per bridge. Degrees $A: 5, B: 3, N: 3, S: 3$.
- (b) **No**: four nodes have odd degree.
- (c) B and N become even, so the walk must **go between A and S** .
- (d) For example **a bridge from A to S** : then every degree is even.

B1	(a) model	1
B1	(a) degrees	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Why this question Each new bridge changes the parity of exactly two places, so you fix odd degrees two at a time.

10 A larger splice

7 marks

- (a) Nodes 1,3,5 have 2 cycle edges and 2 triangle edges: **degree 4**. Nodes 2,4,6 have **degree 2**.
- (b) 1,3,5,1.
- (c) 1,2,3,4,5,6,1.
- (d) 1,2,3,4,5,6,1,3,5,1.
- (e) **9**: six cycle edges and three triangle edges.

B1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
M1	(d)	1
A1	(d)	1
B1	(e)	1

Why this question Here both pieces pass through 1, so the splice is simply “do one, then the other”.

11 Exam style: why the degree conditions work

8 marks

- (a) Each time the circuit passes through a node, it arrives along one edge and leaves along another, using two edges. The start is also left once at the beginning and entered once at the end. Every edge is used once, so **every degree is a sum of 2s**.
- (b) Add a new edge vw . Now every degree is even and the graph is connected, so **there is an Euler circuit**. Start it so the new edge comes last, $v, \dots, w, v$; then **removing that last edge leaves an Euler walk from v to w** .
- (c) Degrees 1: 2, 2: 3, 3: 3, 4: 2, so the odd nodes are **2 and 3**. For example 2,1,3,2,4,3.

E1	(a) two edges per pass	1
E1	(a) the start and end	1
M1	(b) adding vw	1
E1	(b) circuit exists	1
E1	(b) removing the edge	1
B1	(c) odd nodes	1
M1	(c)	1
A1	(c)	1

Common mistake In (a), forgetting the start node. It is left at the very start and entered at the very end, which also uses two edges in total.

12 Exam style: a multigraph with a loop

8 marks

- (a) $A: 2 + 1 = 3$. $B: 2 + 2 = 4$. $C: 2 + 1 + 2 = 5$, since the loop counts twice.
- (b) The multigraph is connected and **exactly A and C are odd**, so there is an Euler walk **from A to C** .
- (c) Call the edges $AB_1, AB_2, BC_1, BC_2, CA$ and the loop ℓ . For example $A \xrightarrow{AB_1} B \xrightarrow{AB_2} A \xrightarrow{CA} C \xrightarrow{\ell} C \xrightarrow{BC_1} B \xrightarrow{BC_2} C$.
- (d) **Yes**: A and C would become even (4 and 6), with the graph still connected.

B1	(a) A and B	1
B1	(a) C with the loop counted twice	1
E1	(b)	1
B1	(b) endpoints	1
M1	(c) starting at A	1
A1	(c) every edge used	1
A1	(c) copies named	1
B1	(d)	1

Why this question With repeated edges, a list of nodes no longer says which edge was used. Naming the edges is what makes the walk precise.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.