

Topic 13 · Sheet A: Questions

Distance and the test for bipartite graphs

Suggested time

About 75 minutes

Total marks

73

Questions

12

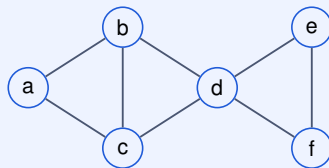
Builds on

Topics 1 to 12

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the distance $d(u, v)$ (the length of a shortest path, or $+\infty$ if there is none), its properties including the triangle inequality, walks and closed walks, and the fact that a graph is bipartite exactly when every closed walk, or equivalently every cycle, has even length.

The parity test. In a connected graph, fix a node v and let L be the nodes at even distance from v and R those at odd distance. The graph is bipartite exactly when no edge has both ends in L or both in R .

The graph G .



1

4 marks

Find these distances in G .

- (a) $d(a, f)$ [1]
- (b) $d(b, e)$ [1]
- (c) $d(c, c)$ [1]
- (d) $d(a, d)$ [1]

2

4 marks

- (a) In C_6 with nodes $1, \dots, 6$ in order, find $d(1, 4)$. [1]
- (b) In the same graph, find $d(1, 5)$. [1]
- (c) In P_5 with nodes $a, \dots, e$ in order, find $d(a, e)$. [1]
- (d) If u and v are in different connected components, what is $d(u, v)$? [1]

3

6 marks

A graph has nodes s, a, b, c, d, e, f and edges $sa, sb, ab, ac, ad, bd, cd, ce, df$.

- (a) Find $d(s, x)$ for every node x . [3]
- (b) Give a shortest path from s to f . [1]
- (c) How many different shortest paths are there from s to f ? [2]

4

6 marks

In a connected graph, $d(u, v) = 3$ and $d(v, w) = 4$.

- (a) Use the triangle inequality to find every possible value of $d(u, w)$. [2]
- (b) Give graphs in which $d(u, w) = 7$ and in which $d(u, w) = 1$. [2]
- (c) Explain why $d(u, w) = 0$ is impossible. [2]

5

6 marks

These parts are about G .

- (a) Give a walk from a to f of length 5 that repeats a node. [1]
- (b) Shorten your walk to a path from a to f . [1]
- (c) Give a closed walk in G of length 5. [2]
- (d) Explain why a bipartite graph has no closed walk of length 5. [2]

6

6 marks

The cube graph Q_3 has the 8 strings $000, 001, \dots, 111$ as nodes, with an edge between two strings that differ in exactly one position.

- (a) Find $d(000, x)$ for every node x . [2]
- (b) Write down the parity test's sets L and R , starting from 000 . [2]
- (c) Explain why every edge joins L to R , and conclude. [2]

7

6 marks

The graph C is C_6 on $1, \dots, 6$ (in order) with the extra edge $\{1, 3\}$.

- (a) Find a cycle of odd length in C . [2]
- (b) Run the parity test from node 1. Find every edge whose ends have distances of the same parity. [2]
- (c) For the edge $\{2, 3\}$, build a closed walk from shortest paths $1 \rightarrow 2$ and $3 \rightarrow 1$ and the edge itself. What is its length? [2]

8

6 marks

Decide whether each graph is bipartite, with a reason.

- (a) C_n , for each $n \geq 3$ [2]
- (b) The complete bipartite graph with parts of sizes 2 and 3 [1]
- (c) The 2×3 grid: top row p, q, r , bottom row s, t, u , edges $pq, qr, st, tu, ps, qt, ru$ [1]
- (d) The wheel: a cycle C_5 plus one more node joined to all five [2]

9

6 marks

The *diameter* of a connected graph is the largest distance between two of its nodes. Find the diameter of each graph.

- (a) C_n [2]
- (b) P_n [1]
- (c) K_n , for $n \geq 2$ [1]
- (d) The cube graph Q_3 [1]
- (e) The complete bipartite graph with parts of sizes 3 and 3 [1]

10

7 marks

A graph has edges 12, 23, 34, 45, 53, 36, 61. Consider the closed walk $W = 1, 2, 3, 4, 5, 3, 6, 1$.

- (a) What is the length of W ? [1]
- (b) Which node, other than the start, does W repeat? [1]
- (c) Split W at that node into two shorter closed walks, and give their lengths. [3]
- (d) Which of the two has odd length? Is it a cycle? [2]

11

8 marks

Exam style. A graph X has nodes $a, \dots, g$ and edges $ab, ac, bd, ce, de, df, eg, fg$.

- (a) Find $d(a, x)$ for every node x . [3]
- (b) Find an edge whose ends are at distances of the same parity. [1]
- (c) Use it to find a cycle of odd length, and conclude whether X is bipartite. [2]
- (d) Show that removing the edge de does not make X bipartite. [2]

12

8 marks

Exam style. The *eccentricity* of a node v is the largest value of $d(v, x)$ over all nodes x .

- (a) Find $d(u, v)$ for every pair of nodes of G . [3]
- (b) Find the eccentricity of each node, and the diameter of G . [2]
- (c) Check the triangle inequality for a, d and f . When does it hold with equality? [1]
- (d) Is G bipartite? Justify your answer. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.