

Topic 13 · Sheet A: Worked answers and mark scheme

Distance and the test for bipartite graphs

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 12

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Distances in G

4 marks

(a) a, b, d, f : **3**. (b) b, d, e : **2**. (c) **0**. (d) a, b, d : **2**.

B1 each part correct

4

Why this question To be sure a distance is 3, check that no path of length 2 exists, for example by looking for common neighbours.

2 Distances in special graphs

4 marks

(a) **3**. (b) **2**, going 1, 6, 5. (c) **4**. (d) $+\infty$.

B1 each part correct

4

Common mistake In (b), going round the long way, 1, 2, 3, 4, 5. A cycle gives two routes; the distance is the shorter.

3 Distances by layers

6 marks

- (a) $s:0; a,b:1; c,d:2; e,f:3$.
- (b) For example s,a,d,f .
- (c) f is reached only from d , and d is reached at distance 2 from both a and b . So there are 2: s,a,d,f and s,b,d,f .

M1	(a) layer by layer from s	1
A1	(a) distances 1 and 2	1
A1	(a) distance 3	1
B1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Building the distance layers one at a time is exactly how a computer finds distances. Each node's distance is one more than that of its nearest neighbour.

4 Using the triangle inequality

6 marks

- (a) $d(u,w) \leq 3 + 4 = 7$, and $4 = d(v,w) \leq d(v,u) + d(u,w) = 3 + d(u,w)$, so $d(u,w) \geq 1$. So $1 \leq d(u,w) \leq 7$, and each value is possible.
- (b) 7: the path u,p,q,v,r,s,t,w , where $d(u,v) = 3$, $d(v,w) = 4$ and $d(u,w) = 7$. 1: the path v,x,y,u,w , where $d(u,v) = 3$, $d(v,w) = 4$ and $d(u,w) = 1$.
- (c) $d(u,w) = 0$ means $u = w$, but then $d(v,w) = d(v,u) = 3 \neq 4$.

M1	(a) upper bound	1
A1	(a) lower bound	1
B1	(b) example for 7	1
B1	(b) example for 1	1
E2	(c)	2

Common mistake Using the triangle inequality only once. Applying it with u in the middle as well gives the lower bound.

5 Walks and closed walks

6 marks

- (a) For example a,b,c,b,d,f .
- (b) Cut out the repeat of b : a,b,d,f .

- (c) For example a, b, c, d, b, a , with edges ab, bc, cd, db, ba : length 5. (A walk may use an edge twice.)
- (d) Every step of a walk in a bipartite graph switches between L and R , so returning to the start takes **an even number of steps**.

B1	(a)	1
B1	(b)	1
M1	(c) a closed walk	1
A1	(c) of length 5	1
E2	(d)	2

Why this question Walks may repeat nodes and edges; paths may not. Removing the loop between two visits to the same node always shortens a walk to a path.

6 The cube is bipartite

6 marks

- (a) $d(000, x)$ is the **number of 1s in x** : 0 for 000; 1 for 001, 010, 100; 2 for 011, 101, 110; 3 for 111.
- (b) $L = \{000, 011, 101, 110\}$, $R = \{001, 010, 100, 111\}$.
- (c) An edge changes one position, so it changes the number of 1s by exactly 1, **switching parity**. So every edge joins L to R and Q_3 is **bipartite**.

M1	(a)	1
A1	(a)	1
B1	(b) L	1
B1	(b) R	1
E1	(c)	1
B1	(c)	1

Why this question Choosing the right invariant, here the number of 1s, checks every edge at once.

7 When the parity test fails

6 marks

- (a) 1,2,3 (a triangle), using the edges 12,23,31.
- (b) Distances from 1: 2,3,6 at 1; 4,5 at 2. The edges {2,3} (both 1) and {4,5} (both 2) have ends of the same parity.
- (c) $1 \rightarrow 2$, then $2 \rightarrow 3$, then $3 \rightarrow 1$: length $1 + 1 + 1 = 3$, which is odd.

B1	(a)	1
B1	(a) edges	1
M1	(b) distances	1
A1	(b) both edges	1
M1	(c)	1
A1	(c)	1

Why this question An edge inside one parity class always gives an odd closed walk: two distances of the same parity plus one edge.

8 Bipartite or not

6 marks

- (a) **Bipartite exactly when n is even**: the only cycle has length n .
- (b) **Bipartite**, by definition.
- (c) **Bipartite**: $\{p, r, t\}$ and $\{q, s, u\}$.
- (d) **Not bipartite**: it contains the odd cycle C_5 (and triangles through the centre).

B1	(a)	1
E1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d)	1
E1	(d)	1

Common mistake Saying the wheel is bipartite because its outer cycle could be split. One odd cycle anywhere is enough to rule it out.

9 Diameters

6 marks

(a) $\lfloor n/2 \rfloor$, between opposite nodes. (b) $n - 1$. (c) **1**. (d) **3**, from 000 to 111. (e) **2**: nodes on the same side are joined through any node on the other side.

M1	(a)	1
A1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d)	1
B1	(e)	1

Why this question Diameter measures how spread out a graph is: the path is as spread out as possible, and the complete graph as compact as possible.

10 Finding an odd cycle in an odd closed walk

7 marks

- (a) **7**.
- (b) **Node 3**.
- (c) **3,4,5,3**, of length **3**, and **1,2,3,6,1**, of length **4**. ($3 + 4 = 7$.)
- (d) **3,4,5,3**. It has no repeated node except its start, so **it is a cycle**, a triangle.

B1	(a)	1
B1	(b)	1
M1	(c) splitting at node 3	1
A1	(c) both walks	1
A1	(c) both lengths	1
B1	(d)	1
E1	(d)	1

Why this question The two lengths add to an odd number, so one must be odd. Repeating the split on the odd one always ends at an odd cycle.

11 Exam style: running the test

8 marks

- (a) **$a: 0; b, c: 1; d, e: 2; f, g: 3$** .
- (b) **de** , with both ends at distance 2. (Also fg , both at 3.)
- (c) Follow **a, b, d** , the edge **de** , then **e, c, a** : **a, b, d, e, c, a** , a cycle of length **5**. So **X is not bipartite**.
- (d) Without **de** , the cycle **a, b, d, f, g, e, c, a** still has odd length **7**.

M1	(a)	1
A1	(a) layers 1 and 2	1
A1	(a) layer 3	1
B1	(b)	1
B1	(c) odd cycle	1
B1	(c) not bipartite	1
M1	(d)	1
A1	(d)	1

Common mistake Deciding “bipartite” after checking only some edges. The test must check every edge; here de and fg both fail.

12 Exam style: all distances in G

8 marks

- (a) Distance 1: the 8 edges. Distance 2: ad, be, bf, ce, cf . Distance 3: ae, af .
- (b) Eccentricities: $a, e, f: 3; b, c, d: 2$. Diameter 3.
- (c) $d(a, f) = 3 \leq d(a, d) + d(d, f) = 2 + 1$. Equality holds **because d lies on a shortest path from a to f** .
- (d) **No**: a, b, c is a triangle, a cycle of odd length.

B1	(a) distance-2 pairs	1
B1	(a) distance-3 pairs	1
B1	(a) all others are edges	1
B1	(b) eccentricities	1
B1	(b) diameter	1
B1	(c)	1
B1	(d)	1
E1	(d)	1

Why this question A table of all distances is how routing software stores a road network. The triangle inequality is what makes going through a middle point never shorter than going direct.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.