

## Topic 12 · Sheet A: Questions

# Vertex covers and König's theorem

Suggested time

**About 75 minutes**

Total marks

**73**

Questions

**12**

Builds on

**Topics 1 to 11**

**How to use this sheet.** Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need vertex covers, alternating paths, the bound  $|M| \leq |S|$  for any matching  $M$  and vertex cover  $S$ , and König's theorem with its construction.

**König's construction.** In a bipartite graph  $(L \cup R, E)$  with a maximum matching  $M$ , let  $U$  be the unmatched nodes of  $L$ , and  $Z$  the nodes reachable from  $U$  by an alternating path (including  $U$  itself). Then  $S = (L \setminus Z) \cup (R \cap Z)$  is a vertex cover with  $|S| = |M|$ .

**1**

4 marks

The path  $P_5$  has nodes  $a, b, c, d, e$  in order.

- (a) Is  $\{b, d\}$  a vertex cover? [1]
- (b) Is  $\{a, c, e\}$  a vertex cover? [1]
- (c) Is  $\{b, c\}$  a vertex cover? [1]
- (d) What is the size of a minimum vertex cover? [1]

**2**

4 marks

- (a) In  $P_5$  from question 1,  $\{ab, cd\}$  is a matching. What does it tell you about every vertex cover? [1]
- (b) Hence explain why  $\{b, d\}$  is a minimum vertex cover. [1]
- (c) Find the size of a maximum matching and of a minimum vertex cover in  $K_4$ . What does this show? [2]

**3**

6 marks

Find the size of a minimum vertex cover of each graph, with a reason.

- (a)  $K_n$  [2]
- (b)  $C_5$  [2]
- (c)  $C_6$  [1]
- (d) The star with one centre joined to 5 other nodes [1]

4

6 marks

- (a) For the triangle  $C_3$ , find a maximum matching and a minimum vertex cover. [2]
- (b) Do the same for  $C_5$ . [2]
- (c) Explain why  $|M| \leq |S|$  still holds in these graphs, even though they are not bipartite. [2]

5

6 marks

In  $P_5$  (nodes  $a, b, c, d, e$  in order) let  $M = \{bc\}$ .

- (a) Is  $(a, b, c)$  an alternating path? Is it an alternating chain? [2]
- (b) Is  $(a, b, c, d, e)$  an alternating path? [2]
- (c) Is the single node  $(a)$  an alternating path? [1]
- (d) Give an alternating chain with respect to  $M$ . [1]

6

6 marks

A bipartite graph has  $L = \{a, b, c\}$  and  $R = \{x, y\}$ , with edges  $ax, bx, cx, cy$ .

- (a) Find a maximum matching  $M$ . [1]
- (b) Write down  $U$ , the unmatched nodes of  $L$ . [1]
- (c) Find  $Z$ , the nodes reachable from  $U$  by alternating paths. [2]
- (d) Find  $S = (L \setminus Z) \cup (R \cap Z)$ . Check that it is a vertex cover of size  $|M|$ . [2]

7

6 marks

A bipartite graph has  $L = \{1, 2, 3, 4\}$  and  $R = \{a, b, c\}$ , with edges  $1a, 2a, 3a, 3b, 4b, 4c$ . Take  $M = \{1a, 3b, 4c\}$ .

- (a) Explain why  $M$  is a maximum matching. [2]
- (b) Find  $Z$ . [2]
- (c) Find  $S$ , and check that it covers every edge. [2]

8

6 marks

A quick way to find a vertex cover: take any matching  $M$  to which no edge can be added, and put *both* endpoints of every edge of  $M$  into  $S$ .

- (a) Explain why this  $S$  is a vertex cover. [2]
- (b) In  $P_5$ , the matching  $\{bc, de\}$  cannot be extended. Which cover does the method give, and how does it compare with the minimum? [2]
- (c) Use  $|M| \leq |S|$  to show that the method always gives a cover at most twice the minimum size. [2]

9

6 marks

The grid graph has top row  $p, q, r$  and bottom row  $s, t, u$ , with edges  $pq, qr, st, tu, ps, qt, ru$ .

- (a) Show that it is bipartite. [1]
- (b) Find a matching with 3 edges. [2]
- (c) Use Kőnig's theorem to find the size of a minimum vertex cover, and give one. [2]
- (d) Explain why no vertex cover has 2 nodes. [1]

10

7 marks

In the grid below, a *line* is a row or a column.

|       | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|-------|
| $r_1$ | 1     | 1     | 0     | 0     |
| $r_2$ | 1     | 0     | 0     | 0     |
| $r_3$ | 1     | 0     | 0     | 0     |
| $r_4$ | 0     | 1     | 1     | 1     |

- (a) Describe a bipartite graph in which a set of 1s with no two in the same line is a matching, and a set of lines containing every 1 is a vertex cover. [1]
- (b) Find the largest number of 1s with no two in the same line. [2]
- (c) Find the smallest number of lines that contain every 1, and give such a set. [2]
- (d) Explain how your answers to (b) and (c) illustrate Kőnig's theorem. [2]

11

8 marks

*Exam style.* A bipartite graph has  $L = \{a, b, c, d, e\}$  and  $R = \{1, 2, 3, 4, 5\}$ , with  $N(a) = \{1, 2\}$ ,  $N(b) = \{1\}$ ,  $N(c) = \{1, 2\}$ ,  $N(d) = \{2, 3, 4\}$  and  $N(e) = \{4, 5\}$ .

- (a) Find a matching of size 4, and explain why no matching has size 5. [2]
- (b) Taking  $M = \{b1, a2, d3, e4\}$ , find  $U$  and  $Z$ . [3]
- (c) Find  $S$  and check that it is a vertex cover. [2]
- (d) State the size of a minimum vertex cover, with a reason. [1]

12

8 marks

*Exam style.*

- (a) For the odd cycle  $C_{2k+1}$ , find the sizes of a maximum matching and a minimum vertex cover. [3]
- (b) Do the same for the even cycle  $C_{2k}$ . [2]
- (c) A bipartite graph on  $n$  nodes has a matching of size  $n/2$ . What is the size of a minimum vertex cover? [1]
- (d) In any graph, suppose a matching  $M$  and a vertex cover  $S$  have  $|M| = |S|$ . Explain why  $M$  is a maximum matching and  $S$  is a minimum vertex cover. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.