

Topic 12 · Sheet A: Worked answers and mark scheme

Vertex covers and König's theorem

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 11

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Checking covers

4 marks

(a) **Yes.** (b) **Yes.** (c) **No:** de is not covered. (d) **2.**

B1 each part correct

4

Why this question A vertex cover must touch every edge. The quickest check is to go along the edges one at a time.

2 A lower bound from a matching

4 marks

(a) **Every vertex cover has at least 2 nodes**, since $|M| \leq |S|$.

(b) $\{b, d\}$ is a cover of size 2, which is **as small as possible**.

(c) A maximum matching has **2** edges and a minimum vertex cover has **3** nodes (leaving out two nodes would leave the edge between them uncovered). So **the two can differ** in a graph that is not bipartite.

B1	(a)	1
B1	(b)	1
B1	(c) both sizes	1
E1	(c) strict inequality possible	1

Why this question A matching proves a lower bound for covers; a cover proves an upper bound for matchings. When the two numbers meet, both are optimal.

3 Minimum covers of special graphs

6 marks

- (a) $n - 1$. Any $n - 1$ nodes cover every edge, but leaving out two nodes leaves the edge between them uncovered.
- (b) **3**. Two nodes cover at most 4 of the 5 edges, and three nodes such as 1, 3, 4 work.
- (c) **3**: every other node, such as 1, 3, 5. Each node covers only 2 of the 6 edges, so fewer is impossible.
- (d) **1**: the centre.

B1	(a)	1
E1	(a)	1
B1	(b)	1
E1	(b)	1
B1	(c)	1
B1	(d)	1

Common mistake Answering $\frac{5}{2}$ rounded down, 2, for C_5 . Each node covers only two edges, so two nodes cover at most four of the five edges.

4 Odd cycles

6 marks

- (a) Maximum matching: **one edge**. Minimum vertex cover: **two nodes**.
- (b) Maximum matching: **2 edges**, such as 12, 34. Minimum vertex cover: **3 nodes**, such as 1, 3, 4.
- (c) Each edge of M needs one of its own endpoints in S , and **no two edges of M share an endpoint**, so they use different nodes of S . This never uses bipartiteness.

B1	(a)	1
B1	(a)	1
B1	(b)	1
B1	(b)	1
E2	(c)	2

Why this question The inequality holds in every graph. Equality is what needs the graph to be bipartite.

5 Alternating paths and chains

6 marks

- (a) **It is an alternating path:** $ab \notin M$, then $bc \in M$. **It is not a chain:** its end c is matched.
- (b) **No:** cd and de are both outside M , so the edges do not alternate.
- (c) **Yes:** paths of length zero are allowed.
- (d) (a, b, c, d) .

B1	(a) path	1
B1	(a) not a chain	1
B1	(b)	1
E1	(b)	1
B1	(c)	1
B1	(d)	1

Why this question Every chain is an alternating path, but not the other way round. König's construction uses alternating paths, which may end anywhere.

6 König's construction, small example

6 marks

- (a) For example $M = \{ax, cy\}$. (R has only two nodes, so no matching is larger.)
- (b) $U = \{b\}$.
- (c) From b : the free edge bx reaches x , and the matching edge xa reaches a . From a there is no new free edge. So $Z = \{b, x, a\}$.
- (d) $S = \{c\} \cup \{x\} = \{c, x\}$. Edges ax, bx, cx contain x and cy contains c , so it is a cover, of size $2 = |M|$.

B1	(a)	1
B1	(b)	1
M1	(c) alternating from U	1
A1	(c)	1
B1	(d) S	1
E1	(d) checked	1

Why this question Inside Z , take the right-hand nodes; outside Z , take the left-hand ones. Each matching edge then contributes exactly one node.

7 König's construction again

6 marks

- (a) $N(\{1, 2\}) = \{a\}$, so at most one of 1, 2 can be matched, and a matching has at most 3 edges. $|M| = 3$.
- (b) $U = \{2\}$. From 2: free edge to a , then matching edge to 1; 1 has no other edges. $Z = \{2, a, 1\}$.
- (c) $S = \{3, 4\} \cup \{a\} = \{3, 4, a\}$. Edges $1a, 2a, 3a$ contain a ; $3b$ contains 3; $4b, 4c$ contain 4.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
E1	(c) every edge checked	1

Common mistake Including 3 in Z because it is adjacent to a . From a right-hand node you may only continue along a *matching* edge, and a 's matching edge goes to 1.

8 A quick cover

6 marks

- (a) If an edge had neither endpoint in S , both ends would be unmatched and it could be added to M . That is ruled out, so **every edge meets S** .
- (b) $\{b, c, d, e\}$, of size 4, while the minimum is 2.
- (c) $|S| = 2|M|$, and $|M| \leq$ the minimum cover size. So $|S| \leq 2 \times$ **minimum**.

E2	(a)	2
B1	(b) cover	1
B1	(b) comparison	1
M1	(c) $ S = 2 M $	1
A1	(c)	1

Why this question Finding a minimum cover is hard in general graphs, but this cheap method is never worse than twice the best.

9 Covering a grid

6 marks

- (a) $L = \{p, r, t\}$, $R = \{q, s, u\}$: every edge joins the two.
- (b) For example $\{ps, qt, ru\}$.
- (c) This matching has size 3, and $|L| = 3$, so it is maximum. By König's theorem the minimum cover has **3** nodes, for example $\{q, s, u\}$.
- (d) The three edges ps, qt, ru share no endpoints, so **any cover needs a different node for each**.

B1	(a)	1
B1	(b)	1
B1	(b)	1
B1	(c) size	1
B1	(c) a cover	1
E1	(d)	1

Why this question In a bipartite graph, one whole side is always a vertex cover. It is a minimum one when the matching saturates that side.

10 Lines and ones in a grid

7 marks

- (a) L is the rows, R the columns, and r_i is joined to c_j when the entry is 1.
- (b) **3**, for example r_1c_2, r_2c_1, r_4c_3 . More is impossible: r_2 and r_3 only have a 1 in column c_1 .
- (c) **3**, for example **column** c_1 , **row** r_1 **and row** r_4 .
- (d) Both numbers are 3: in a bipartite graph **the maximum matching and the minimum vertex cover have the same size**.

B1	(a)	1
B1	(b)	1
E1	(b) why no more	1
B1	(c)	1
B1	(c) a set	1
E2	(d)	2

Why this question König's theorem in this form says: the largest set of 1s with no two in a line equals the fewest lines through all the 1s.

11 Exam style: the full construction

8 marks

- (a) For example $\{b1, a2, d3, e4\}$. No matching has size 5: $N(\{a, b, c\}) = \{1, 2\}$, so one of a, b, c is always unmatched.
- (b) $U = \{c\}$. From c : free edges to 1 and 2; matching edges $1b$ and $2a$; from a the free edge $a1$ leads back to 1; b has no other edges. So $Z = \{c, 1, 2, a, b\}$.
- (c) $S = \{d, e\} \cup \{1, 2\} = \{d, e, 1, 2\}$. Edges at a, b, c go to 1 or 2; edges at d, e contain d or e . **All covered.**
- (d) **4:** S is a cover of size 4, and every cover has at least $|M| = 4$ nodes.

B1	(a) matching	1
E1	(a) Hall's condition fails	1
B1	(b) U	1
M1	(b) alternating steps	1
A1	(b) Z	1
B1	(c) S	1
E1	(c) check	1
B1	(d)	1

Common mistake Putting d in Z because d is adjacent to 2. You can only leave a right-hand node along its matching edge, and 2's matching edge goes to a .

12 Exam style: cycles and certificates

8 marks

- (a) Matching: k , since $2k + 1$ nodes allow at most k disjoint edges, and alternate edges give k . Cover: $k + 1$. Each node covers 2 of the $2k + 1$ edges, so k nodes cover at most $2k$; and $k + 1$ nodes suffice.
- (b) Matching k and cover k , since every other node covers all $2k$ edges.
- (c) $n/2$, by König's theorem: the matching is perfect, so it is maximum.
- (d) For any matching M' , $|M'| \leq |S| = |M|$, so **M is maximum**. For any cover S' , $|S'| \geq |M| = |S|$, so **S is minimum**.

B1	(a) matching	1
B1	(a) cover	1
E1	(a) why k nodes are not enough	1
B1	(b) matching	1
B1	(b) cover	1
B1	(c)	1
E2	(d)	2

Why this question Part (d) is the practical use of these results: once you have a matching and a cover of the same size, you have proved both are optimal without searching any further.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.