

Topic 11 · Sheet A: Questions

Alternating chains and maximum matchings

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 10

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need unmatched nodes, alternating chains, how to enlarge a matching along a chain, the symmetric difference $(M \setminus M^*) \cup (M^* \setminus M)$ of two matchings, and the fact that M is a maximum matching if and only if it has no alternating chain.

Alternating chain. A path $(v_1, w_1, v_2, w_2, \dots, v_k, w_k)$ with distinct nodes, whose endpoints v_1 and w_k are unmatched, and whose edges alternate: $\{v_i, w_i\} \notin M$ and $\{w_i, v_{i+1}\} \in M$. Matching edges are drawn thick.

1

4 marks

The path P_6 has nodes $v_1, \dots, v_6$ in order, and $M = \{v_2v_3, v_4v_5\}$.

- (a) Which nodes are unmatched? [1]
- (b) Is M a maximum matching? [1]
- (c) Write down an alternating chain with respect to M . [1]
- (d) Use it to find a larger matching. [1]

2

4 marks

A graph has nodes a, b, c, d, e and edges ab, bc, cd, da, ae . Let $M = \{ab, cd\}$.

- (a) Is (e, a, b, c) an alternating chain? Explain. [1]
- (b) Which nodes are unmatched? [1]
- (c) Explain why no matching in this graph can have more than 2 edges. [1]
- (d) Does M have an alternating chain? [1]

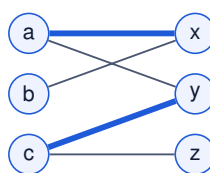
3

6 marks

- (a) An alternating chain has 7 edges. How many of them are in M ? [1]
- (b) Explain why every alternating chain has an odd number of edges. [2]
- (c) Enlarging M along this chain: how many edges leave the matching and how many join it? [2]
- (d) What is the size of the new matching? [1]

4

6 marks



The thick edges form $M = \{ax, cy\}$.

- (a) Which nodes are unmatched? [1]
- (b) Find an alternating chain, and check that it satisfies the definition. [3]
- (c) Write down the matching obtained by enlarging M along your chain. [2]

5

6 marks

The path P_4 has nodes a, b, c, d in order, and $M = \{bc\}$.

- (a) Find an alternating chain with respect to M . [2]
- (b) Write down the larger matching it gives. [2]
- (c) No edge of P_4 can be added to M to give a larger matching. Explain why M is still not a maximum matching. [2]

6

6 marks

The cycle C_6 has nodes $1, \dots, 6$ in order. Let $M = \{12, 34\}$ and $M^* = \{23, 45, 61\}$.

- (a) Write down $(M \setminus M^*) \cup (M^* \setminus M)$. [2]
- (b) Describe this set of edges as a graph. In what order do its edges come from M and M^* ? [2]
- (c) Explain why it is an alternating chain with respect to M . [1]
- (d) What matching do you get by enlarging M along it? [1]

7

6 marks

- (a) In C_5 , with nodes $1, \dots, 5$ in order, let $M = \{12, 34\}$. Which nodes are unmatched? [1]
- (b) Explain why M has no alternating chain. [2]
- (c) What can you conclude about M ? Confirm this by counting nodes. [2]

(d) In C_6 , with nodes $1, \dots, 6$ in order, let $M = \{12, 45\}$. Find an alternating chain. [1]

8

6 marks

A graph has nodes $1, \dots, 8$. Let $M = \{12, 34, 56, 78\}$ and $M^* = \{23, 41, 67, 85\}$ be two matchings in it.

(a) Write down $(M \setminus M^*) \cup (M^* \setminus M)$. [2]

(b) Describe its connected components. [2]

(c) Explain how the components show that $|M| = |M^*|$. [2]

9

6 marks

A bipartite graph has $L = \{a, b, c, d\}$ and $R = \{1, 2, 3, 4\}$, with $N(a) = \{1, 2\}$, $N(b) = \{1\}$, $N(c) = \{2, 3\}$ and $N(d) = \{3, 4\}$. A greedy method takes a, b, c, d in turn and matches each to its smallest neighbour that is still free, if there is one.

(a) Which matching does the greedy method find? [2]

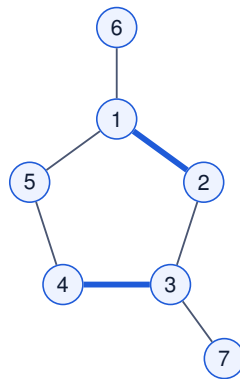
(b) Find an alternating chain with respect to it. [2]

(c) Write down the enlarged matching. [1]

(d) Is the enlarged matching maximum? Explain. [1]

10

7 marks



This graph is a 5-cycle with two extra nodes. The thick edges form $M = \{12, 34\}$.

(a) Which nodes are unmatched? [1]

(b) Find an alternating chain with 3 edges. [2]

(c) Enlarge M along it. [1]

(d) Explain why the result is a maximum matching. [1]

(e) Find a different alternating chain with respect to the original M , with 5 edges. [2]

11

8 marks

Exam style. A bipartite graph has $L = \{a, b, c, d, e\}$ and $R = \{1, 2, 3, 4, 5\}$, with $N(a) = \{1, 2\}$, $N(b) = \{1, 3\}$, $N(c) = \{2\}$, $N(d) = \{3, 4, 5\}$ and $N(e) = \{4\}$. Start with $M = \{a1, b3, d4\}$.

- (a) Which nodes are unmatched? [1]
- (b) Find an alternating chain with one edge, and enlarge M along it. [1]
- (c) Find an alternating chain starting at e for your new matching, and enlarge along it. [3]
- (d) Explain why the final matching is maximum. [1]
- (e) Starting from the empty matching, how many enlargements are needed to reach a matching of size 5? Explain. [2]

12

8 marks

Exam style. A graph contains the path $1, 2, \dots, 8$ and the separate edge $\{9, 10\}$. Let $M = \{23, 45, 67, \{9, 10\}\}$ and $M^* = \{12, 34, 56, 78, \{9, 10\}\}$.

- (a) Write down $|M|$ and $|M^*|$. [1]
- (b) Write down $(M \setminus M^*) \cup (M^* \setminus M)$. [2]
- (c) Describe its components, and explain why one of them is an alternating chain with respect to M . [2]
- (d) Enlarge M along that chain, and compare the result with M^* . [2]
- (e) Explain why the edge $\{9, 10\}$ plays no part. [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.