

Topic 11 · Sheet A: Worked answers and mark scheme

Alternating chains and maximum matchings

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 10

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 A chain along a path

4 marks

(a) v_1 **and** v_6 . (b) **No**. (c) $(v_1, v_2, v_3, v_4, v_5, v_6)$. (d) $\{v_1v_2, v_3v_4, v_5v_6\}$.

B1 each part correct

4

Why this question Swapping which edges of the chain are in the matching keeps every inner node matched once and picks up the two unmatched ends.

2 Checking the definition

4 marks

- (a) **No**: the last node c is matched (by cd), but both ends of a chain must be unmatched.
- (b) e .
- (c) Each edge of a matching uses two nodes, and no node twice, so a matching has at most $\lfloor 5/2 \rfloor = 2$ edges.
- (d) **No**: M is maximum. (Also, there is only one unmatched node, and a chain needs two.)

B1	(a)	1
B1	(b)	1
E1	(c)	1
B1	(d)	1

Common mistake Checking only that the edges alternate. Both endpoints being unmatched is just as much part of the definition.

3 Counting edges in a chain

6 marks

- (a) **3**.
- (b) A chain $(v_1, w_1, \dots, v_k, w_k)$ has k edges outside M and $k - 1$ inside, so $2k - 1$ **edges**, an odd number.
- (c) **3 leave and 4 join**.
- (d) $|M| + 1$.

B1	(a)	1
M1	(b) k outside, $k - 1$ inside	1
A1	(b)	1
B1	(c) leave	1
B1	(c) join	1
B1	(d)	1

Why this question A chain always has one more non-matching edge than matching edge. That extra edge is the $+1$.

4 Enlarging a bipartite matching

6 marks

- (a) **b and z** .
- (b) (b, x, a, y, c, z) . The ends b and z are unmatched. $bx, ay, cz \notin M$ and $xa, yc \in M$, alternating, with distinct nodes.

(c) $\{bx, ay, cz\}$.

B1	(a)	1
M1	(b) starting at an unmatched node	1
A1	(b) a valid chain	1
E1	(b) checking the definition	1
M1	(c) swapping the chain's edges	1
A1	(c)	1

Why this question From b the only edge leads to x , which is matched to a ; from a the free edge leads to y , matched to c ; and c reaches the free node z . Following matched edges back is how you search for a chain.

5 Stuck, but not maximum

6 marks

(a) (a, b, c, d) : $ab \notin M$, $bc \in M$, $cd \notin M$, and a, d are unmatched.

(b) $\{ab, cd\}$.

(c) Every edge of P_4 meets b or c , so none can be added. But M has an alternating chain, so **it is not maximum**: $\{ab, cd\}$ is larger.

B1	(a)	1
E1	(a) checking it	1
M1	(b)	1
A1	(b)	1
E1	(c) no edge can be added	1
E1	(c) still has a chain	1

Common mistake Thinking that a matching you cannot add an edge to must be maximum. Sometimes you have to remove an edge first, which is exactly what a chain does.

6 The symmetric difference

6 marks

- (a) M and M^* share no edges, so it is $\{12, 34, 23, 45, 61\}$.
- (b) **The path** $6, 1, 2, 3, 4, 5$. Its edges $61, 12, 23, 34, 45$ come from M^*, M, M^*, M, M^* .
- (c) It alternates, starting and ending with edges outside M , and its ends 6 and 5 are **unmatched by M** .
- (d) $\{61, 23, 45\} = M^*$.

B1	(a)	1
B1	(a) no common edges	1
B1	(b) a path	1
B1	(b) the pattern	1
E1	(c)	1
B1	(d)	1

Why this question This is the proof that a non-maximum matching has a chain, run on an example: the larger matching M^* shows you where the chain is.

7 When there is no chain

6 marks

- (a) **Only node 5**.
- (b) A chain has two different unmatched endpoints, but **only one node is unmatched**.
- (c) With no chain, **M is a maximum matching**. Directly: 5 nodes allow at most 2 edges, and $|M| = 2$.
- (d) $(3, 4, 5, 6)$: $34 \notin M$, $45 \in M$, $56 \notin M$, with 3 and 6 unmatched.

B1	(a)	1
E2	(b)	2
B1	(c) maximum	1
B1	(c) counting check	1
B1	(d)	1

Why this question “No alternating chain” is a certificate that a matching is maximum. You never have to compare it with every other matching.

8 Two matchings of the same size

6 marks

- (a) No edge is in both, so it is **all eight edges, 12, 34, 56, 78, 23, 41, 67, 85**.
- (b) **Two 4-cycles**: 1, 2, 3, 4 with edges 12, 23, 34, 41, and 5, 6, 7, 8 with edges 56, 67, 78, 85. Each alternates between M and M^* .
- (c) In each cycle the edges alternate, so **half come from each matching**: two from M and two from M^* . So $|M \setminus M^*| = |M^* \setminus M|$, and hence $|M| = |M^*|$.

B1	(a)	1
B1	(a) no common edges	1
B1	(b) two cycles	1
B1	(b) alternating	1
E2	(c)	2

Why this question Even cycles contribute equally to both matchings. Only a path component can make one matching larger than the other.

9 Greedy, then improve

6 marks

- (a) $a \rightarrow 1$; b has no free neighbour; $c \rightarrow 2$; $d \rightarrow 3$. So $\{a1, c2, d3\}$.
- (b) $(b, 1, a, 2, c, 3, d, 4)$, which alternates and joins the unmatched b and 4.
- (c) $\{b1, a2, c3, d4\}$.
- (d) **Yes**: it has $|L| = 4$ edges, and no matching can have more.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
E1	(d)	1

Why this question A single bad early choice ($a \rightarrow 1$) blocks b . The chain undoes it and every later choice along the way.

10 A graph that is not bipartite

7 marks

- (a) **5, 6 and 7**.
- (b) $(7, 3, 4, 5)$: $73 \notin M$, $34 \in M$, $45 \notin M$.

- (c) $\{12, 37, 45\}$.
- (d) There are 7 nodes, so at most 3 edges in any matching, and this one has 3.
- (e) $(6, 1, 2, 3, 4, 5)$: the edges $61, 12, 23, 34, 45$ alternate, and 6 and 5 are unmatched.

B1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
E1	(d)	1
M1	(e)	1
A1	(e)	1

Common mistake Proposing $(6, 1, 5)$. The edge 15 is not in M , so the chain would have two non-matching edges in a row.

11 Exam style: growing to a perfect matching

8 marks

- (a) $c, e, 2$ and 5 .
- (b) $(c, 2)$, since both ends are unmatched. New matching $\{a1, b3, d4, c2\}$.
- (c) e 's only neighbour 4 is matched to d , and d has the free neighbour 5 : $(e, 4, d, 5)$. New matching $\{a1, b3, c2, e4, d5\}$.
- (d) It has $5 = |L|$ edges, the most possible.
- (e) Each enlargement adds exactly one edge, and a chain exists whenever the matching is not maximum, so **exactly 5**.

B1	(a)	1
B1	(b)	1
M1	(c) following $e \rightarrow 4 \rightarrow d$	1
A1	(c) the chain	1
A1	(c) the matching	1
E1	(d)	1
B1	(e)	1
E1	(e)	1

Why this question Repeatedly finding a chain and enlarging is an algorithm for maximum matching. The theorem guarantees it never gets stuck early.

12 Exam style: where the chain comes from

8 marks

- (a) $|M| = 4$, $|M^*| = 5$.
- (b) $\{12, 23, 34, 45, 56, 67, 78\}$.
- (c) **One component, the path $1, 2, \dots, 8$** , with edges alternating $M^*, M, M^*, \dots, M^*$. It starts and ends with edges outside M , and its ends 1 and 8 are not matched by M , so it is an alternating chain.
- (d) $\{12, 34, 56, 78, \{9, 10\}\}$, which **is exactly M^*** .
- (e) It belongs to both M and M^* , so it is in **neither $M \setminus M^*$ nor $M^* \setminus M$** .

B1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c) the path	1
E1	(c) why it is a chain	1
M1	(d)	1
A1	(d)	1
E1	(e)	1

Why this question Since $|M^*| > |M|$, some component of the symmetric difference must have more M^* edges than M edges, and that component is always a chain.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.