

Topic 10 · Sheet A: Questions

Matchings and Hall's theorem

Suggested time

About 75 minutes

Total marks

73

Questions

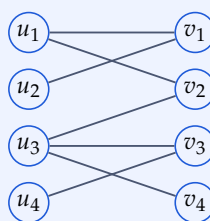
12

Builds on

Topics 1 to 9

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need matchings, the neighbourhood $N(A) = \bigcup_{v \in A} N(v)$ of a set, Hall's condition, and Hall's theorem: a bipartite graph $G = (L \cup R, E)$ has a matching of size $|L|$ if and only if $|N(A)| \geq |A|$ for every $A \subseteq L$.

The graph B . Several questions use this bipartite graph, with $L = \{u_1, \dots, u_4\}$ on the left.



1

4 marks

These parts are about B .

- (a) Is $\{u_1v_1, u_2v_1\}$ a matching? [1]
- (b) Is $\{u_1v_2, u_3v_3\}$ a matching? If so, what is its size? [1]
- (c) Find a matching of size 4. [2]

2

4 marks

Write down each neighbourhood in B .

- (a) $N(\{u_1\})$ [1]
- (b) $N(\{u_1, u_2\})$ [1]
- (c) $N(\{u_3, u_4\})$ [1]
- (d) $N(L)$ [1]

3

6 marks

A bipartite graph has $L = \{a, b, c, d\}$ and $R = \{1, 2, 3, 4\}$, with $N(a) = \{1, 2\}$, $N(b) = \{1\}$, $N(c) = \{1, 2\}$ and $N(d) = \{3, 4\}$.

- (a) Show that Hall's condition fails for $A = \{a, b, c\}$. [2]

- (b) What does Hall's theorem then tell you? [1]
- (c) Find a matching of the largest possible size. [2]
- (d) Add one edge so that a matching of size 4 exists, and give the matching. [1]

4

6 marks

A bipartite graph has $L = \{x, y, z\}$ and $R = \{1, 2, 3\}$, with $N(x) = \{1, 2\}$, $N(y) = \{2, 3\}$ and $N(z) = \{1, 3\}$.

- (a) List every non-empty $A \subseteq L$ with $N(A)$, and check Hall's condition. [3]
- (b) What does Hall's theorem tell you? Find a matching of size 3. [2]
- (c) How many matchings of size 3 are there? [1]

5

6 marks

Four clubs each need a different student as their representative. Chess has members Ana and Ben; Drama has Ben and Cal; Film has Ana and Cal; Music has Ana, Ben, Cal and Dev.

- (a) Describe the bipartite graph that models this. [1]
- (b) Find a representative for each club. [2]
- (c) Dev leaves Music. Use Hall's theorem to show that the clubs can no longer all be represented. [2]
- (d) After Dev leaves, how many clubs can be given different representatives? [1]

6

6 marks

- (a) Use your matching from question 1(c) to explain why $|N(\{u_1, u_3\})| \geq 2$ in B , without computing $N(\{u_1, u_3\})$. [2]
- (b) Explain in general why a matching of size $|L|$ forces $|N(A)| \geq |A|$ for every $A \subseteq L$. [2]
- (c) Explain why a bipartite graph with $|L| > |R|$ cannot have a matching of size $|L|$. [2]

7

6 marks

- (a) In the graph from question 4, show that $|N(A)| \geq |A| + 1$ for every non-empty $A \subsetneq L$. [2]
- (b) Delete z and the node $1 \in N(z)$. Check that the graph left over satisfies Hall's condition, and use this to build a matching of size 3 in the original graph. [2]
- (c) In B , find a non-empty $A \subsetneq L$ with $|N(A)| = |A|$. [2]

8

6 marks

The complete bipartite graph $K_{n,n}$ has $|L| = |R| = n$ and every node of L adjacent to every node of R .

- (a) How many matchings of size 3 does $K_{3,3}$ have? [2]
- (b) How many matchings of size 2 does $K_{3,3}$ have? [2]

(c) How many matchings of size n does $K_{n,n}$ have? [2]

9

6 marks

A bipartite graph has $L = \{1, 2, 3, 4, 5\}$ and $R = \{a, b, c, d, e\}$, with $N(1) = \{a\}$, $N(2) = N(3) = \{a, b\}$, $N(4) = \{c, d, e\}$ and $N(5) = \{c\}$.

(a) Find a set $A \subseteq L$ that breaks Hall's condition. [2]

(b) Find a matching of size 4. [2]

(c) Explain why there is no matching of size 5. [1]

(d) Add one edge so that a matching of size 5 exists. [1]

10

7 marks

Workers $W_1, \dots, W_4$ can do jobs $J_1, \dots, J_5$ as follows: W_1 can do J_1, J_2 ; W_2 can do J_2, J_3 ; W_3 can do J_1, J_3 ; W_4 can do J_1, J_2, J_3 . Each worker gets at most one job and each job at most one worker.

(a) Show that not every worker can be given a job. [2]

(b) What is the largest number of workers that can be given jobs? Give an assignment. [2]

(c) W_4 learns to do J_4 too. Give an assignment of a job to every worker. [2]

(d) Check Hall's condition for $A = L$ after this change. [1]

11

8 marks

Exam style. A bipartite graph has $L = \{\ell_1, \dots, \ell_6\}$ and $R = \{r_1, \dots, r_6\}$. Each ℓ_i is adjacent to r_i and to r_{i+1} , where r_7 means r_1 .

(a) Write down $N(\{\ell_1, \ell_3, \ell_5\})$. [1]

(b) Give two different matchings of size 6. [2]

(c) Show that there are no others. [3]

(d) The edge $\ell_1 r_1$ is removed. How many matchings of size 6 are left? [2]

12

8 marks

Exam style. A bipartite graph has $L = \{a, b, c, d, e\}$ and $R = \{1, 2, 3, 4, 5\}$, with $N(a) = \{1, 2, 3\}$, $N(b) = \{1\}$, $N(c) = \{2\}$, $N(d) = \{1, 2\}$ and $N(e) = \{3, 4, 5\}$.

(a) Find a set $A \subseteq L$ of size 3 that breaks Hall's condition. [2]

(b) Find a matching of size 4. [2]

(c) Use (a) to explain why there is no matching of size 5. [1]

(d) The edge $\{d, 4\}$ is added. Find a matching of size 5. [2]

(e) After adding $\{d, 4\}$, how many matchings of size 5 are there? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.