

Topic 10 · Sheet A: Worked answers and mark scheme

Matchings and Hall's theorem

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 9

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Spotting matchings

4 marks

(a) **No**: both edges have endpoint v_1 .

(b) **Yes**, of size 2.

(c) u_2 can only use v_1 and u_4 only v_3 , which forces u_1v_2 and then u_3v_4 : $\{u_1v_2, u_2v_1, u_3v_4, u_4v_3\}$.

B1	(a)	1
B1	(b)	1
M1	(c) starting with nodes of degree 1	1
A1	(c)	1

Why this question Nodes with only one neighbour have no choice. Matching them first often fixes everything else.

2 Neighbourhoods of sets

4 marks

- (a) $\{v_1, v_2\}$. (b) $\{v_1, v_2\}$. (c) $\{v_2, v_3, v_4\}$. (d) $\{v_1, v_2, v_3, v_4\} = R$.

B1 each part correct

4

Common mistake In (b), adding the sizes $2 + 1 = 3$. The neighbourhood is a union, so shared neighbours count once: v_1 is a neighbour of both.

3 Hall's condition failing

6 marks

- (a) $N(\{a, b, c\}) = \{1, 2\}$, so $|N(A)| = 2 < 3 = |A|$.
(b) There is **no matching of size** $|L| = 4$.
(c) For example $\{a2, b1, d3\}$, of size 3.
(d) For example add $\{c, 3\}$: then $\{a2, b1, c3, d4\}$ is a matching of size 4.

M1	(a) the neighbourhood	1
A1	(a) comparison	1
B1	(b)	1
B1	(c) three edges	1
B1	(c) valid matching	1
B1	(d)	1

Why this question Three clubs sharing only two students can never all be represented. Hall's condition is exactly this kind of bottleneck.

4 Checking every subset

6 marks

- (a) $\{x\} \rightarrow \{1, 2\}$, $\{y\} \rightarrow \{2, 3\}$, $\{z\} \rightarrow \{1, 3\}$, and every pair and L itself $\rightarrow \{1, 2, 3\}$. Every $|N(A)| \geq |A|$, so **Hall's condition holds**.
(b) **A matching of size 3 exists**, for example $\{x1, y2, z3\}$.
(c) **2**: $\{x1, y2, z3\}$ and $\{x2, y3, z1\}$.

M1	(a) all seven subsets	1
A1	(a) neighbourhoods	1
A1	(a) condition holds	1
B1	(b)	1
B1	(b) a matching	1
B1	(c)	1

Why this question With $|L| = n$ there are $2^n - 1$ non-empty subsets to check, so checking Hall's condition directly is only practical for small graphs. Its real value is in proofs.

5 Club representatives

6 marks

- (a) L is the set of clubs, R the set of students, and a club is joined to each of its members.
- (b) For example **Chess–Ana, Drama–Ben, Film–Cal, Music–Dev**.
- (c) Now $N(L) = \{\text{Ana, Ben, Cal}\}$ has size $3 < 4 = |L|$. **Hall's condition fails, so no matching of size 4 exists.**
- (d) **3**, for example Chess–Ana, Drama–Ben, Film–Cal.

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) choosing $A = L$	1
A1	(c)	1
B1	(d)	1

Common mistake Checking only single clubs. Each club alone still has at least two members; the problem only shows up when you look at all four together.

6 The easy direction

6 marks

- (a) The matching sends $u_1 \mapsto v_2$ and $u_3 \mapsto v_4$. These partners are **two different nodes, both in $N(\{u_1, u_3\})$** .
- (b) Each node of A is matched to a neighbour of itself, and **no two nodes share a partner**. So $N(A)$ contains $|A|$ different nodes.
- (c) Take $A = L$: then $|N(L)| \leq |R| < |L|$, so **Hall's condition fails**. Or directly: a matching uses a different node of R for each node of L , and there are not enough.

M1	(a) partners lie in $N(A)$	1
A1	(a) they are distinct	1
E2	(b)	2
E2	(c)	2

Why this question This direction of Hall's theorem is the easy one. The hard direction, that the condition is enough, is what the slack and tight cases in the proof are for.

7 Slack and tight sets

6 marks

- (a) Singletons have neighbourhoods of size $2 = 1 + 1$, and pairs have neighbourhood $\{1, 2, 3\}$, of size $3 = 2 + 1$. **So the slack case applies.**
- (b) Left: x with $N = \{2\}$ and y with $N = \{2, 3\}$. Then $|N(\{x\})| = 1$, $|N(\{y\})| = 2$ and $|N(\{x, y\})| = 2$, so **Hall's condition holds**. Match $x2$, $y3$, then add $z1$: $\{x2, y3, z1\}$.
- (c) $A = \{u_1, u_2\}$: $N(A) = \{v_1, v_2\}$, of size 2.

M1	(a) singletons and pairs	1
A1	(a)	1
M1	(b) checking the smaller graph	1
A1	(b) the matching	1
B1	(c) a tight set	1
B1	(c) its neighbourhood	1

Why this question In the slack case you can match any node to any neighbour and still be safe. In the tight case you must first match A to $N(A)$ exactly, then deal with the rest.

8 Counting matchings in $K_{n,n}$

6 marks

- (a) Send the three nodes of L to different nodes of R : $3! = 6$.
- (b) Choose the two nodes of L ($\binom{3}{2} = 3$ ways), then their partners in order ($3 \cdot 2 = 6$ ways): **18**.
- (c) $n!$, one for each way of pairing L with R .

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake In (b), answering $\binom{3}{2} \cdot \binom{3}{2} = 9$. The partners must be assigned to particular nodes of L , so their order matters.

9 A graph with a bottleneck

6 marks

- (a) $A = \{1, 2, 3\}$: $N(A) = \{a, b\}$, of size $2 < 3$.
- (b) For example $\{1a, 2b, 4d, 5c\}$.
- (c) Hall's condition fails, by (a), so **Hall's theorem rules out a matching of size 5**.

(d) For example add $\{3, e\}$: then $\{1a, 2b, 3e, 4d, 5c\}$ works.

M1	(a)	1
A1	(a)	1
B1	(b) four edges	1
B1	(b) valid	1
E1	(c)	1
B1	(d)	1

Why this question Look for a group of left nodes that only know a few right nodes. That is where a matching gets stuck.

10 Assigning jobs

7 marks

- (a) $N(\{W_1, W_2, W_3, W_4\}) = \{J_1, J_2, J_3\}$, of size $3 < 4$. **Hall's condition fails.**
- (b) **3**, for example W_1J_1, W_2J_2, W_3J_3 .
- (c) For example $W_1J_1, W_2J_2, W_3J_3, W_4J_4$.
- (d) $N(L) = \{J_1, J_2, J_3, J_4\}$, of size $4 = |L|$.

M1	(a)	1
A1	(a)	1
B1	(b) size	1
B1	(b) assignment	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Common mistake Concluding from (d) alone that the new graph has a full matching. Hall's condition needs every subset A ; here (c) shows the matching directly, which settles it.

11 Exam style: a cyclic graph

8 marks

- (a) $\{r_1, r_2, r_3, r_4, r_5, r_6\} = R$.
- (b) $\ell_i r_i$ **for all** i , and $\ell_i r_{i+1}$ **for all** i (with $\ell_6 r_1$).
- (c) Consider ℓ_1 . If $\ell_1 r_1$ is used, then ℓ_6 cannot use r_1 , so $\ell_6 r_6$; then $\ell_5 r_5$, and so on. Everything is forced: the first matching. If $\ell_1 r_2$ is used, then ℓ_2 cannot use r_2 , so $\ell_2 r_3$; then $\ell_3 r_4$, and so on: the second matching. **So there are exactly two.**
- (d) Only the case $\ell_1 r_2$ is left: **1**.

B1	(a)	1
B1	(b) each matching	2
M1	(c) two cases for ℓ_1	1
A1	(c) first case forced	1
A1	(c) second case forced	1
M1	(d)	1
A1	(d)	1

Why this question Splitting into cases on one node and following the forced choices is how you count matchings by hand.

12 Exam style: finding the obstruction

8 marks

- (a) $A = \{b, c, d\}$: $N(A) = \{1, 2\}$, of size $2 < 3$.
- (b) For example $\{a3, b1, c2, e4\}$.
- (c) b, c, d all need different partners from $\{1, 2\}$, which is impossible, so **at most two of them are matched**, and a matching has size at most 4.
- (d) $\{a3, b1, c2, d4, e5\}$.
- (e) b must take 1 and c must take 2, so d must take 4, a must take 3, and e is left with 5: **1**.

M1	(a)	1
A1	(a)	1
B1	(b) four edges	1
B1	(b) valid	1
E1	(c)	1
M1	(d)	1
A1	(d)	1
B1	(e)	1

Common mistake Searching only among nodes of small degree. Here a has three neighbours, but the trouble is the three nodes that only know 1 and 2 between them.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.