

Topic 9 · Sheet A: Questions

Degrees, connectivity and isomorphism

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

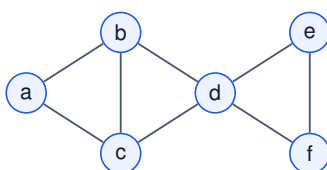
Topics 1 to 8

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need degrees and degree sequences, the fact that $\sum_v \deg(v) = 2|E|$, graphical sequences, connected graphs and connected components, and isomorphism.

Notation. A *simple* graph has no multi-edges and no loops. A sequence is *graphical* if it is the degree sequence of some simple graph. In a multigraph, a loop adds 2 to the degree of its node.

1

4 marks



- (a) What is the degree of d ? [1]
- (b) Write down the degree sequence in non-decreasing order. [2]
- (c) Check that the degrees add up to $2|E|$. [1]

2

4 marks

- (a) What is the degree of each node of K_n ? [1]
- (b) What is the degree of each node of C_n ? [1]
- (c) Write down the degree sequence of P_n , for $n \geq 2$. [1]
- (d) In a multigraph, node v has a loop and one other edge (v, w) . What is $\deg(v)$? [1]

3

6 marks

- (a) A graph has 10 nodes, each of degree 3. How many edges does it have? [2]
- (b) Can a graph have 7 nodes, each of degree 3? Explain. [2]
- (c) A graph has 12 edges and every node has degree 4. How many nodes does it have? [2]

4

6 marks

Decide whether each sequence is graphical. If it is, draw a simple graph with that degree sequence; if not, explain why.

- (a) 1, 1, 2, 3 [1]
- (b) 2, 2, 2, 2 [1]
- (c) 3, 3, 3, 3 [1]
- (d) 0, 2, 2, 3, 3 [2]
- (e) 1, 3, 3, 3 [1]

5

6 marks

A graph has nodes $1, \dots, 9$ and edges $\{1, 2\}, \{2, 3\}, \{1, 3\}, \{4, 5\}, \{6, 7\}, \{7, 8\}, \{6, 9\}$.

- (a) List the node sets of its connected components. [3]
- (b) How many connected components are there? [1]
- (c) What is the smallest number of edges you must add to make the graph connected? [2]

6

6 marks

The graph H has nodes $a, \dots, h$. It consists of the triangle a, b, c , the edge $\{c, d\}$, the triangle d, e, f , and the edges $\{f, g\}$ and $\{g, h\}$.

- (a) Is H connected? Give a path from a to h . [2]
- (b) List every edge whose removal would disconnect H . [3]
- (c) How many connected components are left if $\{c, d\}$ and $\{f, g\}$ are both removed? [1]

7

6 marks

G_1 has nodes $1, \dots, 5$ and edges $12, 23, 34, 45, 51, 13$.

G_2 has nodes $a, \dots, e$ and edges ab, ac, ad, bc, ce, de .

- (a) Write down the degree sequences of G_1 and G_2 . [2]
- (b) Find an isomorphism $f : \{1, \dots, 5\} \rightarrow \{a, \dots, e\}$. [3]
- (c) Check that f maps every edge of G_1 to an edge of G_2 . [1]

8

6 marks

- (a) Show that C_6 , and the graph made of two separate triangles, have the same degree sequence. Explain why they are not isomorphic. [3]
- (b) Name one more property that isomorphic graphs must share. [1]
- (c) Are C_4 and the complete bipartite graph with $|L| = |R| = 2$ isomorphic? If so, give an isomorphism. [2]

9

6 marks

- (a) How many nodes of odd degree does the graph in question 1 have? [1]
- (b) At a party, some pairs of people shake hands. Explain why the number of people who shake an odd number of hands must be even. [2]
- (c) A graph has 30 edges. Nine of its nodes have degree 4 and every other node has degree 3. How many nodes does it have? [3]

10

7 marks

- (a) How many simple graphs on 3 nodes are there, up to isomorphism? Draw one of each. [2]
- (b) How many simple graphs on 4 nodes with exactly 3 edges are there, up to isomorphism? Draw one of each. [3]
- (c) Explain why no two of your graphs in (b) are isomorphic. [2]

11

8 marks

Exam style. Consider the degree sequence 3, 3, 2, 2, 2.

- (a) How many edges does a graph with this degree sequence have? [1]
- (b) Draw a simple graph with this degree sequence in which the two nodes of degree 3 are adjacent. [2]
- (c) Draw a simple graph with this degree sequence in which the two nodes of degree 3 are not adjacent. [3]
- (d) Explain why your graphs in (b) and (c) are not isomorphic. [2]

12

8 marks

Exam style. P is the prism: two triangles 1, 2, 3 and 4, 5, 6 with the extra edges 14, 25, 36. Q is the complete bipartite graph with $L = \{x_1, x_2, x_3\}$ and $R = \{y_1, y_2, y_3\}$. S is the cycle C_6 on $1, \dots, 6$ together with the three edges 14, 25, 36.

- (a) Write down the degree sequence of each graph. [1]
- (b) How many edges does each graph have? [1]
- (c) How many triangles does P contain? How many does Q contain? [2]
- (d) Hence explain why P and Q are not isomorphic. [1]
- (e) Show that S is isomorphic to Q by giving an isomorphism. [3]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.