

Topic 9 · Sheet A: Worked answers and mark scheme

Degrees, connectivity and isomorphism

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 8

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Degrees in a drawing

4 marks

- (a) **4**: its neighbours are b, c, e, f .
- (b) $a: 2, b: 3, c: 3, d: 4, e: 2, f: 2$, so **2, 2, 2, 3, 3, 4**.
- (c) The sum is 16, and the graph has 8 edges: **$16 = 2 \cdot 8$** .

B1	(a)	1
M1	(b) each degree	1
A1	(b) in order	1
B1	(c)	1

Why this question Counting edges and adding degrees should always agree. It is a quick check that you have read the drawing correctly.

2 Degrees in special graphs

4 marks

(a) $n - 1$. (b) 2. (c) $1, 1, 2, \dots, 2$, with $n - 2$ twos. (d) 3: the loop counts twice.

B1 each part correct

4

Common mistake Counting a loop once. The degree is the number of times v appears as an endpoint, and a loop (v, v) has v as both endpoints.

3 The handshake fact

6 marks

(a) $2|E| = 10 \cdot 3 = 30$, so $|E| = 15$.

(b) No: the degrees would add up to 21, which is odd, but the sum must be $2|E|$.

(c) $4n = 2 \cdot 12 = 24$, so $n = 6$.

M1	(a)	1
A1	(a)	1
B1	(b) no	1
E1	(b) odd sum	1
M1	(c)	1
A1	(c)	1

Why this question Every edge is counted twice when you add the degrees, once from each end. That one idea answers all three parts.

4 Graphical or not?

6 marks

- (a) **Not graphical**: the sum 7 is odd.
- (b) **Graphical**: C_4 .
- (c) **Graphical**: K_4 .
- (d) **Graphical**: an isolated node a , with d and e adjacent to each other and both adjacent to b and c . Edges de, db, dc, eb, ec .
- (e) **Not graphical**: each node of degree 3 must be adjacent to all three other nodes, so the last node would have degree 3, not 1.

B1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d) graphical	1
B1	(d) a valid graph	1
B1	(e) with reason	1

Common mistake In (e), concluding “graphical” because the sum 10 is even. An even sum is necessary, but not enough for a simple graph.

5 Connected components

6 marks

- (a) $\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8, 9\}$.
- (b) **3**.
- (c) **2**, for example $\{3, 4\}$ and $\{5, 6\}$. Each new edge joins at most two components into one.

B1	(a) each component	3
B1	(b)	1
B1	(c)	1
E1	(c) each edge merges at most two components	1

Why this question Components are the equivalence classes of “there is a path between them”. Adding an edge can only merge two classes at a time.

6 Edges that hold a graph together

6 marks

- (a) **Yes**. For example a, c, d, f, g, h .
- (b) $\{c, d\}, \{f, g\}$ **and** $\{g, h\}$. Each edge of a triangle can be avoided by going round the other two sides, so removing it leaves H connected.

(c) **3:** $\{a, b, c\}$, $\{d, e, f\}$ and $\{g, h\}$.

B1	(a) yes	1
B1	(a) a valid path	1
B1	(b) all three edges	1
B1	(b) no extra edges	1
E1	(b) triangle edges can be avoided	1
B1	(c)	1

Why this question An edge disconnects the graph exactly when it lies on no cycle. You will meet these edges again as bridges.

7 Finding an isomorphism

6 marks

- (a) Both are $2, 2, 2, 3, 3$. In G_1 , nodes 1 and 3 have degree 3; in G_2 , nodes a and c do.
- (b) Both graphs are a triangle sharing an edge with a 4-cycle: $1, 2, 3$ and $1, 3, 4, 5$ in G_1 ; a, b, c and a, c, e, d in G_2 . So $1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto e, 5 \mapsto d$.
- (c) $12 \mapsto ab, 23 \mapsto bc, 34 \mapsto ce, 45 \mapsto ed, 51 \mapsto da, 13 \mapsto ac$: **all edges of G_2** .

B1	(a) each sequence	2
M1	(b) degree-3 nodes mapped to degree-3 nodes	1
M1	(b) using the structure	1
A1	(b)	1
B1	(c)	1

Common mistake Mapping $4 \mapsto d$ and $5 \mapsto e$. Then $34 \mapsto cd$, which is not an edge. Every edge must be checked.

8 Same degrees, different graphs

6 marks

- (a) Both have six nodes of degree 2: $2, 2, 2, 2, 2, 2$. But C_6 **is connected and two triangles are not**, and an isomorphism maps paths to paths, so it preserves connectedness.
- (b) For example **the number of edges, the number of triangles, or the number of components**.
- (c) **Yes**. With $L = \{a, b\}$ and $R = \{x, y\}$, the edges are ax, ay, bx, by , which form the cycle a, x, b, y . So $a \mapsto v_1, x \mapsto v_2, b \mapsto v_3, y \mapsto v_4$.

B1	(a) same sequence	1
E2	(a) connectedness differs, and is preserved	2
B1	(b)	1
B1	(c) yes	1
B1	(c) an isomorphism	1

Why this question Equal degree sequences are necessary for isomorphism, but not enough. To prove two graphs differ, find a property one has and the other lacks.

9 Odd degrees

6 marks

- (a) **2**: b and c .
- (b) People are nodes and handshakes are edges. The degrees add up to $2|E|$, which is even, so **the number of odd degrees is even**.
- (c) With k nodes of degree 3: $9 \cdot 4 + 3k = 60$, so $k = 8$ and there are **17 nodes**.

B1	(a)	1
E1	(b) model as a graph	1
E1	(b) even sum forces an even count	1
M1	(c) degree sum = 60	1
A1	(c) $k = 8$	1
A1	(c) 17	1

Common mistake Answering 8 in (c). That is the number of degree-3 nodes; the question asks for all the nodes.

10 Small graphs up to isomorphism

7 marks

- (a) **4**: no edges, one edge, a path P_3 , and K_3 .
- (b) **3**: a triangle plus an isolated node; the path P_4 ; and the star, one centre joined to the other three.
- (c) Their degree sequences differ: **0, 2, 2, 2, 1, 1, 2, 2** and **1, 1, 1, 3**. Isomorphic graphs have the same degree sequence.

B1	(a) count	1
B1	(a) drawings	1
B1	(b) count	1
B2	(b) drawings. <i>B1 for two correct</i>	2
M1	(c) comparing an invariant	1
A1	(c)	1

Why this question Graphs with 3 edges on 3 nodes can only be K_3 ; on 4 nodes there is room for three different shapes.

11 Exam style: one sequence, two graphs

8 marks

- (a) $(3 + 3 + 2 + 2 + 2)/2 = 6$.
- (b) Let u, v have degree 3. For example uv, ux, uy, vx, vz, yz .
- (c) Join both u and v to all of x, y, z : ux, uy, uz, vx, vy, vz . This is the complete bipartite graph with parts $\{u, v\}$ and $\{x, y, z\}$.
- (d) An isomorphism maps degree-3 nodes to degree-3 nodes, and adjacent nodes to adjacent nodes. In (b) the two degree-3 nodes are adjacent and in (c) they are not, **so no isomorphism exists**.

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) u, v share all neighbours	1
A1	(c)	1
A1	(c) degrees checked	1
E2	(d)	2

Why this question A degree sequence does not determine a graph. Looking at how the high-degree nodes connect is often the quickest way to tell graphs apart.

12 Exam style: three cubic graphs

8 marks

- (a) All three: 3, 3, 3, 3, 3, 3.
- (b) All three: 9.
- (c) P : 2, the triangles 1, 2, 3 and 4, 5, 6 (any other three nodes include a missing edge such as 24). Q : 0, since it is bipartite.
- (d) Isomorphisms map triangles to triangles, so isomorphic graphs have the same number of triangles. $2 \neq 0$.
- (e) In S every edge joins an odd number to an even one, and each odd node is adjacent to all three even nodes. So $1 \mapsto x_1, 3 \mapsto x_2, 5 \mapsto x_3, 2 \mapsto y_1, 4 \mapsto y_2, 6 \mapsto y_3$, which sends the 9 edges of S to the 9 edges of Q .

B1	(a)	1
B1	(b)	1
B1	(c) P	1
B1	(c) Q	1
E1	(d)	1
M1	(e) odd and even nodes as the parts	1
A1	(e) an isomorphism	1
A1	(e) all edges checked	1

Common mistake Deciding $S \cong P$ because both are drawn with a hexagon. Drawings can mislead; count triangles or look for a bipartition.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.