

Topic 8 · Sheet A: Worked answers and mark scheme

Introduction to graphs: special graphs, subgraphs and bipartite graphs

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 7

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Reading the graph

4 marks

(a) $|V| = 6$, $|E| = 8$. (b) $N(d) = \{b, c, e, f\}$. (c) $N[a] = \{a, b, c\}$. (d) **No**.

B1 each part correct

4

Common mistake Leaving a out of $N[a]$. The closed neighbourhood includes the node itself; the open one $N(a)$ does not.

2 Special graphs

4 marks

(a) $\binom{5}{2} = 10$. (b) **7**. (c) **5**. (d) $\binom{n}{2} = \frac{n(n-1)}{2}$.

B1 each part correct

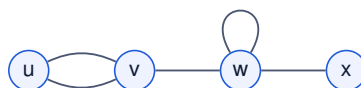
4

Why this question A cycle has as many edges as nodes; a path has one fewer. K_n has an edge for every pair of nodes.

3 Multi-edges and loops

6 marks

- (a) **Yes:** e_1 and e_2 have the same endpoints.
- (b) e_4 .
- (c) Nodes u, v, w, x in a row, with two edges between u and v , one each for $v-w$ and $w-x$, and a loop at w :



- (d) Remove e_2 (**or** e_1) **and** e_4 . What is left is the path P_4 : u, v, w, x .

B1	(a) with reason	1
B1	(b)	1
B1	(c) the two parallel edges	1
B1	(c) the rest, including the loop	1
B1	(d) edges removed	1
B1	(d) P_4	1

Why this question Naming the edges $e_1, \dots, e_5$ is what lets a multigraph have two different edges with the same endpoints.

4 Subgraphs and induced subgraphs

6 marks

- (a) **Yes, a subgraph:** its nodes and edges all belong to G . **Not induced:** $\{b, c\}$ has both ends in $\{a, b, c\}$ but is missing.
- (b) $\{b, c\}, \{b, d\}, \{c, d\}, \{d, e\}$.
- (c) The edges of G inside $\{a, b, c\}$ are ab, ac, bc , and any subset of them gives a subgraph: $2^3 = 8$. Only the one with all three edges is induced: **1**.

B1	(a) subgraph	1
B1	(a) not induced, with the missing edge	1
M1	(b) edges with both ends in the set	1
A1	(b)	1
B1	(c) 8	1
B1	(c) 1	1

Why this question Choosing the nodes fixes the induced subgraph completely. A general subgraph can also leave edges out.

5 Paths, cycles and cliques

6 marks

- (a) For example a, b, d, f . **No path of length 2**: it would need a common neighbour, but $N(a) = \{b, c\}$ and $N(f) = \{d, e\}$ do not meet.
- (b) For example a, b, d, c , with edges $\{a, b\}, \{b, d\}, \{d, c\}, \{c, a\}$.
- (c) **3**, since there is no clique of size 4 (for example $\{a, b, c, d\}$ is missing $\{a, d\}$). The triangles are $\{a, b, c\}, \{b, c, d\}$ **and** $\{d, e, f\}$.

B1	(a) a valid path	1
E1	(a) no common neighbour	1
B1	(b) a valid cycle	1
B1	(b) its edges	1
B1	(c) size 3	1
B1	(c) all three triangles	1

Common mistake Listing a, b, c, d, a as a 4-cycle. Check every consecutive pair: $\{a, d\}$ is not an edge.

6 Bipartite or not

6 marks

- (a) $L = \{v_1, v_3\}, R = \{v_2, v_4\}$.
- (b) In any split of $\{v_1, v_2, v_3\}$ into two parts, **two of the three nodes are in the same part**, and in K_3 those two are joined by an edge.
- (c) Take $L = \{1, 3, 5\}, R = \{2, 4, 6\}$. The cycle edges join consecutive numbers, so one odd and one even, and $\{1, 4\}$ also joins odd to even.
- (d) **No**: 1, 2, 3 would form a triangle, and a triangle cannot be split as in (b).

B1	(a)	1
M1	(b) two nodes in one part	1
A1	(b) they are adjacent	1
B1	(c) parts	1
E1	(c) every edge checked, including $\{1, 4\}$	1
B1	(d)	1

Why this question To show a graph is bipartite, give the parts and check every edge. To show it is not, find a small piece, like a triangle, that cannot be split.

7 Counting edges and graphs

6 marks

- (a) $\binom{n}{2}$.
- (b) Each of the 3 possible edges is in or out: $2^3 = 8$. In general $2^{\binom{n}{2}}$.
- (c) Every node of L joined to every node of R : $3 \cdot 4 = 12$.
- (d) With $|L| = k$ and $|R| = 10 - k$, the maximum is $k(10 - k)$, largest at $k = 5$: **25**.

B1	(a)	1
B1	(b) $\{1, 2, 3\}$	1
B1	(b) general	1
B1	(c)	1
M1	(d) maximising $k(10 - k)$	1
A1	(d)	1

Common mistake In (d), answering $\binom{10}{2} = 45$. A bipartite graph has no edges inside L or inside R , so it has far fewer.

8 Counting subgraphs of K_4

6 marks

- (a) Any 3 nodes form a triangle: $\binom{4}{3} = 4$. In K_n : $\binom{n}{3}$.
- (b) Choose the middle node (4 ways) and its two ends from the other three ($\binom{3}{2} = 3$ ways): **12**.
- (c) A 4-cycle uses all four nodes. Fixing node 1, there are $3! = 6$ orders of the others, and each cycle is counted twice (once in each direction): **3**.

B1	(a) K_4	1
B1	(a) K_n	1
M1	(b) choose the middle, then the ends	1
A1	(b)	1
M1	(c) dividing by 2 for direction	1
A1	(c)	1

Why this question A subgraph is a set of edges, so the same path or cycle walked in a different direction is the same subgraph.

9 Induced subgraphs of C_5

6 marks

- (a) The path P_3 : 1, 2, 3.
- (b) **Two nodes and no edges**, since 1 and 3 are not adjacent.

- (c) The **5** sets of three consecutive nodes, such as $\{1, 2, 3\}$, induce P_3 . The other $\binom{5}{3} - 5 = 5$ sets, such as $\{1, 2, 4\}$, induce **one edge plus an isolated node**.
- (d) **No**: no three nodes of C_5 are pairwise adjacent.

B1	(a)	1
B1	(b)	1
B1	(c) 5 sets	1
M1	(c) counting the others	1
A1	(c) what they induce	1
B1	(d)	1

Why this question Choosing nodes of a cycle splits it into pieces. Counting what each choice induces is good practice for the definitions.

10 Checking a bipartition

7 marks

- (a) $L = \{p, r, t, w\}$ **and** $R = \{q, s, u\}$. Every edge, $pq, qr, ps, qt, ru, st, tu, uw$, has one end in each.
- (b) **No**. p, q, t would then be pairwise adjacent. p and q are in different parts, and so are q and t , so p and t are in the same part, but they are adjacent.
- (c) **Yes**: $p \in L$ and $u \in R$, so the same parts still work.

B2	(a) correct parts. <i>B1 for one node misplaced</i>	2
E1	(a) every edge checked	1
B1	(b) no	1
E1	(b) the triangle p, q, t	1
B1	(c) yes	1
E1	(c)	1

Common mistake Concluding “not bipartite” because your first choice of parts fails. You must show that *no* split works, for example with a triangle.

11 Exam style: a graph from a rule

8 marks

- (a) $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}$ and $\{1, 4\}, \{2, 5\}, \{3, 6\}$: $|E| = 8$.
- (b) $N(2) = \{1, 3, 5\}$ and $N[4] = \{1, 3, 4, 5\}$.
- (c) Both 1 and 3 are odd, so every edge joins an odd number to an even one: $L = \{1, 3, 5\}$, $R = \{2, 4, 6\}$.
- (d) For example **1, 2, 5, 4**: the edges 12, 25, 54, 41 all exist. A 3-cycle would need three pairwise adjacent nodes, but two of them would lie in the same part, which has no edges inside it.

B1	(a) the edges	1
B1	(a) $ E = 8$	1
B1	(b) $N(2)$	1
B1	(b) $N[4]$	1
B1	(c) parts	1
E1	(c) differences are odd	1
B1	(d) a 4-cycle	1
E1	(d) no triangle	1

Why this question Spotting the parity of $|i - j|$ proves bipartiteness for every edge at once, with no checking one by one.

12 Exam style: a friendship graph

8 marks

- (a) $V = \{A, B, C, D, E\}$, $E = \{AB, AC, AD, BC, DE\}$.
- (b) $\{A, B, C\}$, size 3. No clique of size 4 exists, since E has only one friend and B, C are not friends with D .
- (c) C, A, D, E , of length 3.
- (d) The edges inside are AC, AD and DE : **the path** P_4, C, A, D, E .
- (e) **2**: add BD and CD to make $\{A, B, C, D\}$ a clique. Any 4 nodes including E need at least 3 new edges, since E has only one friend.

B1	(a)	1
B1	(b)	1
E1	(b) no larger clique	1
B1	(c)	1
M1	(d) edges with both ends in the set	1
A1	(d)	1
B1	(e)	1
E1	(e) why fewer is impossible	1

Common mistake In (e), forgetting to check all choices of four people. The best choice avoids E , who has only one friend.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.