

Topic 7 · Sheet A: Questions

Generating functions and solving recurrences

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 6

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the generating function $G(x) = \sum_{k \geq 0} a_k x^k$ of a sequence, the geometric series, differentiating a series, partial fractions, and the method for solving a recurrence with a generating function.

Useful facts. $\frac{1}{1-ax} = \sum_{k \geq 0} a^k x^k$ for $|ax| < 1$. To solve a recurrence that holds for $n \geq 1$, multiply it by x^n and sum over $n \geq 1$. Always check your formula against the first few terms.

1

4 marks

Write the generating function of each sequence $a_0, a_1, a_2, \dots$ in closed form.

- (a) $1, 1, 1, 1, \dots$ [1]
- (b) $1, 2, 4, 8, \dots$ [1]
- (c) $1, -1, 1, -1, \dots$ [1]
- (d) $0, 1, 1, 1, \dots$ [1]

2

4 marks

Find a formula for a_k , the coefficient of x^k , in each generating function.

- (a) $\frac{1}{1-5x}$ [1]
- (b) $\frac{3}{1-x}$ [1]
- (c) $\frac{1}{1-x^2}$ [1]
- (d) $\frac{2}{1-3x} - \frac{1}{1-x}$ [1]

3

6 marks

- (a) By differentiating $\sum_{k \geq 0} x^k = \frac{1}{1-x}$, show that $\frac{1}{(1-x)^2} = \sum_{k \geq 0} (k+1)x^k$. [3]
- (b) Hence find the generating function of $0, 1, 2, 3, \dots$ (that is, $a_k = k$). [1]
- (c) Find the coefficient of x^k in $\frac{1}{(1-2x)^2}$. [2]

4

6 marks

Let $G(x) = \frac{1+x}{(1-2x)(1-3x)}$.

- (a) Find constants A and B with $G(x) = \frac{A}{1-2x} + \frac{B}{1-3x}$. [3]
- (b) Hence find the coefficient a_k of x^k in $G(x)$. [2]
- (c) Check a_0 and a_1 by expanding $G(x)$ directly. [1]

5

6 marks

Solve each recurrence using a generating function.

- (a) $a_n = 3a_{n-1}$ for $n \geq 1$, with $a_0 = 2$. [3]
- (b) $a_n = 2a_{n-1} + 1$ for $n \geq 1$, with $a_0 = 0$. [3]

6

6 marks

Let $a_n = 3a_{n-1} + 2$ for $n \geq 1$, with $a_0 = 1$, and let $G(x)$ be its generating function.

- (a) Show that $G(x) - 1 = 3xG(x) + \frac{2x}{1-x}$. [2]
- (b) Hence write $G(x)$ as a single fraction. [1]
- (c) Use partial fractions to split $G(x)$. [2]
- (d) Find a_n , and check a_1 . [1]

7

6 marks

Let $G(x)$ be the generating function of $a_0, a_1, a_2, \dots$

- (a) Write down the generating function of $a_k = k + 1$. [1]
- (b) Write down the generating function of $a_k = 3^k + 1$. [1]
- (c) Write down the generating function of $a_k = 2^{k+1}$. [1]
- (d) Which sequence has generating function $\frac{x^2}{1-x}$? [1]
- (e) Which sequence has generating function $xG(x)$? [1]
- (f) Write the generating function of $a_1, a_2, a_3, \dots$ in terms of $G(x)$. [1]

8

6 marks

Let $a_n = 2a_{n-1} + 2^n$ for $n \geq 1$, with $a_0 = 1$.

- (a) Find an equation for its generating function $G(x)$. [2]
- (b) Show that $G(x) = \frac{1}{(1-2x)^2}$. [1]
- (c) Use question 3(c) to find a_n . [2]
- (d) Check a_1 and a_2 against the recurrence. [1]

9

6 marks

Let $a_n = a_{n-1} + n$ for $n \geq 1$, with $a_0 = 0$. You may use $\frac{1}{(1-x)^3} = \sum_{k \geq 0} \frac{(k+1)(k+2)}{2} x^k$.

- (a) Show that its generating function is $G(x) = \frac{x}{(1-x)^3}$. [3]
- (b) Hence find a_n . [2]
- (c) Explain why your answer was to be expected. [1]

10

7 marks

Let $a_n = 4a_{n-1} + 3^n$ for $n \geq 1$, with $a_0 = 1$.

- (a) Show that $G(x)(1-4x) = 1 + \frac{3x}{1-3x}$. [3]
- (b) Hence show that $G(x) = \frac{1}{(1-3x)(1-4x)}$. [1]
- (c) Find a_n using partial fractions. [2]
- (d) Check a_1 and a_2 against the recurrence. [1]

11

8 marks

Exam style. Let $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 4$.

- (a) Multiply by x^n and sum over $n \geq 2$ to show that $G(x) - 1 - 4x = 5x(G(x) - 1) - 6x^2G(x)$. [3]
- (b) Hence show that $G(x) = \frac{1-x}{(1-2x)(1-3x)}$. [2]
- (c) Find a_n . [2]
- (d) Check a_2 . [1]

12

8 marks

Exam style. Let $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 6$.

- (a) Show that its generating function is $G(x) = \frac{1}{(1-3x)^2}$. [3]
- (b) Hence find a_n . [2]
- (c) Check a_2 and a_3 against the recurrence. [1]

(d) Give a_n in Θ notation.

[2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.