

Topic 7 · Sheet A: Worked answers and mark scheme

Generating functions and solving recurrences

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 6

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Closed forms

4 marks

(a) $\frac{1}{1-x}$. (b) $\frac{1}{1-2x}$. (c) $\frac{1}{1+x}$. (d) $\frac{x}{1-x}$.

B1 each part correct

4

Why this question All four are the geometric series with a different ratio, or shifted by a factor of x .

2 Reading off coefficients

4 marks

(a) $a_k = 5^k$. (b) $a_k = 3$. (c) $a_k = 1$ if k is even, 0 if k is odd. (d) $a_k = 2 \cdot 3^k - 1$.

B1 each part correct

4

Why this question Part (d) is the last step of every recurrence on this sheet: once $G(x)$ is a sum of simple fractions, you read off a_k term by term.

3 Differentiating a series

6 marks

- (a) Differentiate both sides: $\frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}$ and $\frac{d}{dx} \sum_{k \geq 0} x^k = \sum_{k \geq 1} kx^{k-1}$. Replacing k by $k+1$: $\frac{1}{(1-x)^2} = \sum_{k \geq 0} (k+1)x^k$.
- (b) Multiply by x : $\frac{x}{(1-x)^2}$.
- (c) Replace x by $2x$: $\frac{1}{(1-2x)^2} = \sum (k+1)2^k x^k$, so the coefficient is $(k+1)2^k$.

M1	(a) differentiating both sides	1
M1	(a) the $k = 0$ term vanishes	1
A1	(a) reindexing	1
B1	(b)	1
M1	(c) substituting $2x$	1
A1	(c)	1

Common mistake In (c), answering $k+1$. Substituting $2x$ for x turns x^k into $2^k x^k$, so the coefficient picks up 2^k .

4 Partial fractions

6 marks

- (a) $A(1-3x) + B(1-2x) = 1+x$. Constants: $A+B=1$. Coefficient of x : $-3A-2B=1$. Solving, $A=-3$, $B=4$.
- (b) $a_k = 4 \cdot 3^k - 3 \cdot 2^k$.
- (c) $\frac{1}{(1-2x)(1-3x)} = (1+2x+\dots)(1+3x+\dots) = 1+5x+\dots$, and multiplying by $1+x$ gives $1+6x+\dots$. From (b), $a_0 = 4-3=1$ and $a_1 = 12-6=6$. **They agree.**

M1	(a) clearing denominators	1
M1	(a) comparing coefficients	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1

Why this question Comparing coefficients is usually quickest. You can also substitute $x = \frac{1}{2}$ and $x = \frac{1}{3}$ to kill one term at a time.

5 Two first-order recurrences

6 marks

- (a) $\sum_{n \geq 1} a_n x^n = 3x \sum_{n \geq 0} a_n x^n$ gives $G(x) - 2 = 3xG(x)$, so $G(x) = \frac{2}{1-3x}$ and $a_n = 2 \cdot 3^n$.
- (b) $G(x) - 0 = 2xG(x) + \sum_{n \geq 1} x^n = 2xG(x) + \frac{x}{1-x}$. So $G(x) = \frac{x}{(1-x)(1-2x)} = \frac{1}{1-2x} - \frac{1}{1-x}$, and $a_n = 2^n - 1$.

M1	(a) equation for G	1
M1	(a) solving for G	1
A1	(a)	1
M1	(b) equation for G	1
M1	(b) partial fractions	1
A1	(b)	1

Common mistake Writing $\sum_{n \geq 1} a_n x^n = G(x)$. The sum starts at $n = 1$, so it is $G(x) - a_0$.

6 A recurrence with a constant

6 marks

- (a) Multiply by x^n and sum over $n \geq 1$: $\sum_{n \geq 1} a_n x^n = G(x) - 1$, $\sum_{n \geq 1} 3a_{n-1} x^n = 3xG(x)$ and $\sum_{n \geq 1} 2x^n = \frac{2x}{1-x}$.
- (b) $G(x)(1-3x) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$, so $G(x) = \frac{1+x}{(1-x)(1-3x)}$.
- (c) $A(1-3x) + B(1-x) = 1+x$ gives $A+B=1$ and $3A+B=-1$, so $A=-1$, $B=2$:
 $G(x) = \frac{2}{1-3x} - \frac{1}{1-x}$.
- (d) $a_n = 2 \cdot 3^n - 1$. Check: $a_1 = 3 \cdot 1 + 2 = 5 = 6 - 1$.

M1	(a)	1
A1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Why this question This is the full method in four steps: equation for G , solve for G , partial fractions, read off a_n .

7 Building generating functions

6 marks

- (a) $\frac{1}{(1-x)^2}$. (b) $\frac{1}{1-3x} + \frac{1}{1-x}$. (c) $\frac{2}{1-2x}$. (d) $0, 0, 1, 1, 1, \dots$
 (e) $0, a_0, a_1, a_2, \dots$ (f) $\frac{G(x) - a_0}{x}$.

B1 each part correct

6

Why this question Multiplying by x shifts a sequence right; subtracting a_0 and dividing by x shifts it left. These two moves are exactly what turn a recurrence into an equation for $G(x)$.

8 A repeated factor

6 marks

- (a) $G(x) - 1 = 2xG(x) + \sum_{n \geq 1} 2^n x^n = 2xG(x) + \frac{2x}{1-2x}$.
 (b) $G(x)(1-2x) = 1 + \frac{2x}{1-2x} = \frac{1}{1-2x}$, so $G(x) = \frac{1}{(1-2x)^2}$.
 (c) $a_n = (n+1)2^n$.
 (d) $a_1 = 2 + 2 = 4 = 2 \cdot 2$ and $a_2 = 8 + 4 = 12 = 3 \cdot 4$. **Both agree.**

M1	(a)	1
A1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Common mistake Trying partial fractions $\frac{A}{1-2x} + \frac{B}{1-2x}$. A repeated factor cannot be split that way; use the expansion of $\frac{1}{(1-2x)^2}$ directly.

9 Adding up 1 to n

6 marks

- (a) $G(x) - 0 = xG(x) + \sum_{n \geq 1} nx^n = xG(x) + \frac{x}{(1-x)^2}$. So $G(x)(1-x) = \frac{x}{(1-x)^2}$, giving $G(x) = \frac{x}{(1-x)^3}$.
 (b) The coefficient of x^n in $\frac{x}{(1-x)^3}$ is the coefficient of x^{n-1} in $\frac{1}{(1-x)^3}$: $\frac{n(n+1)}{2}$. So $a_n = \frac{n(n+1)}{2}$.
 (c) Unrolling, $a_n = 1 + 2 + \dots + n$, which is **the arithmetic series**.

M1	(a) the sum of nx^n	1
M1	(a) collecting G terms	1
A1	(a)	1
M1	(b) shifting the index by one	1
A1	(b)	1
E1	(c)	1

Why this question Generating functions reproduce a formula you already know here, which is a good check that the method is working before you trust it on harder recurrences.

10 Two different powers

7 marks

- (a) $G(x) - 1 = 4xG(x) + \sum_{n \geq 1} 3^n x^n = 4xG(x) + \frac{3x}{1-3x}$. Rearranging gives the result.
- (b) $1 + \frac{3x}{1-3x} = \frac{1}{1-4x}$. Divide by $1-4x$.
- (c) $A(1-4x) + B(1-3x) = 1$ gives $A+B=1$ and $4A+3B=0$, so $A=-3$, $B=4$. So $a_n = 4 \cdot 4^n - 3 \cdot 3^n = 4^{n+1} - 3^{n+1}$.
- (d) $a_1 = 4 + 3 = 7 = 16 - 9$ and $a_2 = 28 + 9 = 37 = 64 - 27$. **Both agree.**

M1	(a) left-hand side $G - 1$	1
M1	(a) $4xG$	1
A1	(a) geometric sum	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Common mistake Writing $\sum_{n \geq 1} 3^n x^n = \frac{1}{1-3x}$. Starting at $n = 1$ removes the constant term: it is $\frac{1}{1-3x} - 1 = \frac{3x}{1-3x}$.

11 Exam style: a second-order recurrence

8 marks

- (a) $\sum_{n \geq 2} a_n x^n = G - 1 - 4x$. $\sum_{n \geq 2} 5a_{n-1} x^n = 5x \sum_{m \geq 1} a_m x^m = 5x(G-1)$. $\sum_{n \geq 2} 6a_{n-2} x^n = 6x^2 G$. This gives the equation.
- (b) $G(1-5x+6x^2) = 1+4x-5x = 1-x$, and $1-5x+6x^2 = (1-2x)(1-3x)$.
- (c) $A(1-3x) + B(1-2x) = 1-x$: $A+B=1$, $3A+2B=1$, so $A=-1$, $B=2$. $a_n = 2 \cdot 3^n - 2^n$.
- (d) $a_2 = 5 \cdot 4 - 6 \cdot 1 = 14$, and $2 \cdot 9 - 4 = 14$. **It agrees.**

M1	(a) left side	1
M1	(a) shifting the middle sum	1
A1	(a)	1
M1	(b) factorising	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Why this question The same method handles two earlier terms; you just need two starting values and to sum from $n = 2$. The roots of $1 - 5x + 6x^2$ give the bases 2 and 3.

12 Exam style: a repeated root

8 marks

- (a) Summing from $n \geq 2$: $G - 1 - 6x = 6x(G - 1) - 9x^2G$. So $G(1 - 6x + 9x^2) = 1 + 6x - 6x = 1$, and $1 - 6x + 9x^2 = (1 - 3x)^2$.
- (b) Replacing x by $3x$ in $\frac{1}{(1-x)^2} = \sum (k+1)x^k$: $a_n = (n+1)3^n$.
- (c) $a_2 = 36 - 9 = 27 = 3 \cdot 9$ and $a_3 = 162 - 54 = 108 = 4 \cdot 27$. **Both agree.**
- (d) $a_n = \Theta(n \cdot 3^n)$, since $(n+1)3^n$ is between $n \cdot 3^n$ and $2n \cdot 3^n$ for $n \geq 1$.

M1	(a) sums for each term	1
M1	(a) collecting	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
B1	(d)	1
E1	(d)	1

Why this question A repeated root brings in an extra factor of n . Without it, $a_n = c \cdot 3^n$ could not match both $a_0 = 1$ and $a_1 = 6$.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.