

Topic 2 · Sheet A: Questions

Asymptotic notation for sums

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topic 1

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the rules for sums: pulling out constants, geometric and arithmetic series, bounding parts of a sum, integration bounds, and sums of the form $\sum i \cdot a^i$.

Useful facts. $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=0}^n x^i = \frac{x^{n+1}-1}{x-1}$ for $x \neq 1$, and $\sum_{i=1}^n \frac{1}{i} = \Theta(\log n)$. If f is non-decreasing and $a < b$ are integers, then $\int_{a-1}^b f(x) dx \leq \sum_{i=a}^b f(i) \leq \int_a^{b+1} f(x) dx$.

1

4 marks

Give the simplest $h(n)$ with each sum equal to $\Theta(h(n))$. No working is needed.

(a) $\sum_{i=1}^n i$ [1]

(b) $\sum_{i=1}^n 2^i$ [1]

(c) $\sum_{i=1}^n (1/3)^i$ [1]

(d) $\sum_{i=1}^n 1^i$ [1]

2

4 marks

(a) Find a closed form for $\sum_{i=1}^n (3i + 2)$. [2]

(b) Hence give its Θ class. [1]

(c) Give the Θ class of $\sum_{i=1}^n (n - i)$. [1]

3

6 marks

(a) Show that $\sum_{i=0}^n 3^i \leq \frac{3}{2} \cdot 3^n$, and hence that $\sum_{i=0}^n 3^i = \Theta(3^n)$. [3]

(b) Show that $\sum_{i=1}^n (0.9)^i \leq 9$ for every n , and hence give its Θ class. [2]

(c) Explain why the ratio $x = 1$ needs a separate case. [1]

4

6 marks

Let $S(n) = \sum_{i=1}^n i^2$. Do *not* use a closed form for $S(n)$.

- (a) Show that $S(n) = O(n^3)$. [2]
- (b) By keeping only the terms with $i \geq n/2$, show that $S(n) \geq \frac{1}{8}n^3$. [3]
- (c) State the conclusion in Θ notation. [1]

5

6 marks

Use $\sum_{i=1}^n \frac{1}{i} = \Theta(\log n)$ to give the Θ class of each sum, with a reason.

- (a) $\sum_{i=1}^n \frac{3n}{i}$ [2]
- (b) $\sum_{i=1}^n \frac{n+i}{i}$ [2]
- (c) $\sum_{i=1}^n \left(\frac{1}{i} + \frac{1}{2^i} \right)$ [2]

6

6 marks

- (a) Use the integration bounds to show that $\frac{2}{3}n^{3/2} \leq \sum_{i=1}^n \sqrt{i} \leq \frac{2}{3}((n+1)^{3/2} - 1)$. [3]
- (b) Hence give the Θ class of $\sum_{i=1}^n \sqrt{i}$. [1]
- (c) Use the integration bounds to show that $\sum_{i=1}^n i^3 \geq \frac{1}{4}n^4$. [2]

7

6 marks

- (a) Give the Θ class of $\sum_{i=1}^n i \cdot 2^i$, $\sum_{i=1}^n i \cdot (1/2)^i$ and $\sum_{i=1}^n i \cdot 1^i$. [3]
- (b) Show directly that $n \cdot 2^n \leq \sum_{i=1}^n i \cdot 2^i \leq n \cdot 2^{n+1}$. [2]
- (c) Explain how (b) confirms your first answer in (a). [1]

8

6 marks

Each fragment runs a constant-time step once per iteration of its inner loop. Write the number of steps as a sum and give its Θ class.

- (a) **for** $i = 1$ **to** n : **for** $j = 1$ **to** i^2 : step [2]
- (b) **for** $i = 1$ **to** n : $j \leftarrow 1$; **while** $j \leq i$: step; $j \leftarrow 2j$ [2]
- (c) **for** $i = 1$ **to** n : **for** $j = 1$ **to** $\lfloor n/i \rfloor$: step [2]

9

6 marks

Let $T(n) = \sum_{i=1}^n i \log_2 i$.

- (a) Show that $T(n) = O(n^2 \log n)$. [2]
- (b) By bounding part of the sum, show that $T(n) \geq \frac{1}{8}n^2 \log_2 n$ for $n \geq 4$. [3]
- (c) State the conclusion in Θ notation. [1]

10

7 marks

Let $n = 2^k$ for an integer $k \geq 0$, so that $\log_2 n = k$.

- (a) Evaluate $\sum_{i=0}^k 2^i$ in terms of n , and give its Θ class. [2]
- (b) Evaluate $\sum_{i=0}^k \frac{n}{2^i}$ in terms of n , and give its Θ class. [2]
- (c) Evaluate $\sum_{i=0}^k n$, and give its Θ class. [2]
- (d) In each of (a), (b) and (c), say which terms decide the size of the sum. [1]

11

8 marks

Exam style. Let $S(n) = \sum_{i=1}^n (i^2 + 3^i)$.

- (a) Give the Θ class of $\sum_{i=1}^n i^2$. [1]
- (b) Find a closed form for $\sum_{i=1}^n 3^i$. [2]
- (c) Hence give the Θ class of $S(n)$, with a reason. [2]
- (d) Let $T(n) = \sum_{i=1}^n \sum_{j=1}^i 3^j$. Find a closed form for $T(n)$ and its Θ class. [3]

12

8 marks

Exam style. Let $S(n) = \sum_{i=1}^n (n - i + 1) \cdot 2^i$.

- (a) Substitute $k = n - i + 1$ to show that $S(n) = 2^{n+1} \sum_{k=1}^n \frac{k}{2^k}$. [2]
- (b) Using the rule for $\sum i \cdot a^i$, explain why $\sum_{k=1}^n \frac{k}{2^k}$ is $\Theta(1)$. [2]
- (c) Hence show that $S(n) = \Theta(2^n)$, giving an explicit lower bound. [3]
- (d) Which grows faster, $S(n)$ or $\sum_{i=1}^n i \cdot 2^i$? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.