

Topic 2 · Sheet A: Worked answers and mark scheme

Asymptotic notation for sums

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topic 1

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Quick sums

4 marks

(a) $\Theta(n^2)$. (b) $\Theta(2^n)$. (c) $\Theta(1)$. (d) $\Theta(n)$.

B1 each part correct

4

Why this question These are the three geometric cases (ratio above, below and equal to 1) plus the arithmetic series. Every later question reduces to one of them.

2 Arithmetic series

4 marks

(a) $3 \sum i + \sum 2 = \frac{3n(n+1)}{2} + 2n = \frac{3n^2 + 7n}{2}$.

(b) $\Theta(n^2)$.

(c) The terms are $n-1, n-2, \dots, 0$, which sum to $\frac{n(n-1)}{2}$: $\Theta(n^2)$.

M1	(a) splitting the sum	1
A1	(a)	1
B1	(b)	1
B1	(c)	1

Why this question Part (c) is the inner-loop count of many algorithms: “for each i , do $n - i$ steps”.

3 Geometric series

6 marks

- (a) $\sum_{i=0}^n 3^i = \frac{3^{n+1} - 1}{2} \leq \frac{3^{n+1}}{2} = \frac{3}{2} \cdot 3^n$. Also the sum is at least its last term 3^n . So $3^n \leq \sum \leq \frac{3}{2} \cdot 3^n$, and the sum is $\Theta(3^n)$.
- (b) $\sum_{i=1}^n 0.9^i \leq \sum_{i=1}^{\infty} 0.9^i = \frac{0.9}{1 - 0.9} = 9$, and the sum is at least 0.9. So it is $\Theta(1)$.
- (c) The formula $\frac{x^{n+1} - 1}{x - 1}$ divides by zero when $x = 1$. Then every term is 1, so the sum is n (or $n + 1$), which is neither of the other cases.

M1	(a) closed form	1
A1	(a) upper bound	1
B1	(a) lower bound from the last term	1
M1	(b) comparing with the infinite sum	1
A1	(b)	1
E1	(c)	1

Why this question An increasing geometric sum is a constant times its *largest* term; a decreasing one is a constant times its *first* term.

4 Bounding parts of a sum

6 marks

- (a) Each of the n terms is at most n^2 , so $S(n) \leq n \cdot n^2 = n^3$. So $S(n) = O(n^3)$.
- (b) There are at least $\frac{n}{2}$ terms with $i \geq \frac{n}{2}$, and each is at least $(\frac{n}{2})^2$. So $S(n) \geq \frac{n}{2} \cdot \frac{n^2}{4} = \frac{1}{8}n^3$.
- (c) $S(n) = \Theta(n^3)$.

M1	(a) each term $\leq n^2$	1
A1	(a)	1
M1	(b) at least $n/2$ terms kept	1
M1	(b) each kept term $\geq n^2/4$	1
A1	(b)	1
B1	(c)	1

Common mistake Lower-bounding every term by the smallest one, 1. That only gives $S(n) \geq n$, which is far too weak. Keep the top half, where every term is big.

5 Pulling out constants

6 marks

- (a) $3n$ does not depend on i , so the sum is $3n \sum \frac{1}{i} = 3n \cdot \Theta(\log n) = \Theta(n \log n)$.
- (b) $\frac{n+i}{i} = \frac{n}{i} + 1$, so the sum is $n \sum \frac{1}{i} + n = \Theta(n \log n) + \Theta(n) = \Theta(n \log n)$.
- (c) $\sum \frac{1}{2^i} = \Theta(1)$, so the sum is $\Theta(\log n) + \Theta(1) = \Theta(\log n)$.

M1	(a) pulling out $3n$	1
A1	(a)	1
M1	(b) splitting $\frac{n+i}{i}$	1
A1	(b)	1
M1	(c) splitting into two sums	1
A1	(c)	1

Common mistake Treating n as a variable of the sum. Only i changes as you sum; n is a constant factor and can be pulled out.

6 Integration bounds

6 marks

- (a) $\sqrt{x}$ is non-decreasing. With $a = 1$, $b = n$: $\int_0^n \sqrt{x} dx = \frac{2}{3}n^{3/2}$ and $\int_1^{n+1} \sqrt{x} dx = \frac{2}{3}((n+1)^{3/2} - 1)$. **This gives the bounds.**
- (b) The upper bound is at most $\frac{2}{3}(2n)^{3/2} = O(n^{3/2})$, so $\Theta(n^{3/2})$.
- (c) x^3 is non-decreasing for $x \geq 0$, so $\sum_{i=1}^n i^3 \geq \int_0^n x^3 dx = \frac{1}{4}n^4$.

M1	(a) using non-decreasing $\sqrt{x}$ with the right limits	1
A1	(a) lower integral	1
A1	(a) upper integral	1
B1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Integrals turn sums with no neat closed form into something you can compute. The lower integral starts at $a - 1$ and the upper one ends at $b + 1$.

7 Sums of $i \cdot a^i$

6 marks

- (a) $\Theta(n \cdot 2^n)$, $\Theta(1)$ and $\Theta(n^2)$.
- (b) Lower bound: the last term alone is $n \cdot 2^n$. Upper bound: each $i \leq n$, so the sum is at most $n \sum_{i=1}^n 2^i \leq n(2^{n+1} - 2) \leq n \cdot 2^{n+1}$.
- (c) The bounds are 1 and 2 times $n \cdot 2^n$, so the sum is $\Theta(n \cdot 2^n)$.

B1	(a) each class	3
B1	(b) lower bound	1
B1	(b) upper bound	1
E1	(c)	1

Common mistake Answering $\Theta(2^n)$ for the first sum. The factor i is close to n for the big terms, so it survives as a factor n .

8 Counting loop steps

6 marks

- (a) $\sum_{i=1}^n i^2 = \Theta(n^3)$.
- (b) For each i , j takes the values $1, 2, 4, \dots$ up to i : that is $\lfloor \log_2 i \rfloor + 1$ steps. The total is at most $n(\log_2 n + 1)$, and the top half of the terms gives at least $\frac{n}{2} \log_2 \frac{n}{2}$. So $\Theta(n \log n)$.
- (c) $\sum_{i=1}^n \lfloor \frac{n}{i} \rfloor$ lies between $n \sum \frac{1}{i} - n$ and $n \sum \frac{1}{i}$, so it is $\Theta(n \log n)$.

M1	(a) the sum	1
A1	(a)	1
M1	(b) $\lfloor \log_2 i \rfloor + 1$ steps for each i	1
A1	(b)	1
M1	(c) the harmonic sum	1
A1	(c)	1

Why this question Doubling a counter gives logarithmic loops, and “ n/i steps for each i ” gives the harmonic sum. Both appear constantly in runtime analysis.

9 A sum with a logarithm

6 marks

- (a) Each term is at most $n \log_2 n$, and there are n of them, so $T(n) \leq n^2 \log_2 n$.
- (b) Keep the at least $\frac{n}{2}$ terms with $i \geq \frac{n}{2}$. Each is at least $\frac{n}{2} \log_2 \frac{n}{2}$. So $T(n) \geq \frac{n^2}{4} (\log_2 n - 1)$. For $n \geq 4$, $\log_2 n - 1 \geq \frac{1}{2} \log_2 n$, giving $T(n) \geq \frac{1}{8} n^2 \log_2 n$.
- (c) $T(n) = \Theta(n^2 \log n)$.

M1	(a)	1
A1	(a)	1
M1	(b) top half of the terms	1
M1	(b) $\log_2 \frac{n}{2} = \log_2 n - 1$	1
A1	(b)	1
B1	(c)	1

Common mistake Writing $\log_2 \frac{n}{2} = \frac{1}{2} \log_2 n$. In fact $\log_2 \frac{n}{2} = \log_2 n - 1$: dividing inside a log subtracts outside it.

10 Three shapes of sum

7 marks

- (a) $2^{k+1} - 1 = 2n - 1$, so $\Theta(n)$.
- (b) $n \left(2 - \frac{1}{2^k}\right) = 2n - 1$, so $\Theta(n)$.
- (c) $(k+1)n = n(\log_2 n + 1)$, so $\Theta(n \log n)$.
- (d) In (a) the **last term** dominates, in (b) the **first term** dominates, and in (c) **all terms are equal**, so the number of terms matters.

M1	(a) geometric sum	1
A1	(a)	1
M1	(b) pulling out n	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
E1	(d)	1

Why this question These three sums are exactly the three cases of the Master Theorem, which you will meet soon: the cost is decided by the top level, the bottom level, or all levels equally.

11 Exam style: polynomial plus exponential

8 marks

- (a) $\Theta(n^3)$.
- (b) $\frac{3^{n+1} - 3}{2}$.
- (c) $\Theta(n^3) + \Theta(3^n)$ and $n^3 = o(3^n)$, so $S(n) = \Theta(3^n)$.
- (d) The inner sum is $\frac{3^{i+1}-3}{2}$. So $T(n) = \frac{1}{2} \sum_{i=1}^n 3^{i+1} - \frac{3n}{2} = \frac{1}{2} \cdot \frac{3^{n+2}-9}{2} - \frac{3n}{2} = \frac{3^{n+2} - 6n - 9}{4}$, which is $\Theta(3^n)$.

B1	(a)	1
M1	(b) geometric formula	1
A1	(b)	1
M1	(c) dropping the smaller part	1
A1	(c)	1
M1	(d) inner sum	1
A1	(d) closed form	1
A1	(d) $\Theta(3^n)$	1

Common mistake Expecting the double sum to be bigger by a factor of n . The inner sums grow geometrically, so the last one dominates and the total stays $\Theta(3^n)$. Check: $T(2) = 3 + 12 = 15$.

12 Exam style: reversing a sum

8 marks

- (a) With $k = n - i + 1$, $i = n - k + 1$, and k runs from n down to 1. So $S(n) = \sum_{k=1}^n k \cdot 2^{n-k+1} = 2^{n+1} \sum_{k=1}^n \frac{k}{2^k}$.
- (b) It is $\sum k \cdot a^k$ with $a = \frac{1}{2} < 1$, which is $\Theta(1)$. (In fact the sum is always less than 2.)
- (c) Upper bound: $S(n) \leq 2^{n+1} \cdot 2 = 2^{n+2}$. Lower bound: the term $i = n$ alone gives $S(n) \geq 2^n$. So $S(n) = \Theta(2^n)$.
- (d) $\sum i \cdot 2^i$, which is $\Theta(n \cdot 2^n)$.

M1	(a) substitution	1
A1	(a)	1
B1	(b)	1
E1	(b) ratio $\frac{1}{2} < 1$	1
M1	(c) upper bound	1
A1	(c) lower bound	1
A1	(c) conclusion	1
B1	(d)	1

Why this question The big factor 2^i is attached to the *small* weight $n - i + 1$. Reversing the order makes this visible and turns the sum into one you know.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.