

Topic 1 · Sheet A: Questions

Asymptotic notation: O , Ω , Θ , o and ω

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Nothing earlier

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the definitions of O , Ω , Θ , o and ω (with constants c and N), the limit tests, and the simplification rules.

Notation. All functions take positive values on the positive integers. $g(n) = O(f(n))$ means there are constants $c, N > 0$ with $g(n) \leq c \cdot f(n)$ for all $n > N$, and similarly for the others. $\log n$ means $\log_2 n$ unless a base is given.

1

4 marks

Decide whether each statement is true or false. No justification is needed.

- (a) $3n + 7 = O(n)$ [1]
- (b) $n^2 = O(n)$ [1]
- (c) $n = \Omega(\log n)$ [1]
- (d) $5n^3 + n = \Theta(n^3)$ [1]

2

4 marks

For each statement, give constants c and N that make the definition work, and check that they do.

- (a) $4n + 9 = O(n)$ [2]
- (b) $n^2 - 3n = \Omega(n^2)$ [2]

3

6 marks

Let $f(n) = 2n^2 + 5n + 3$.

- (a) Find a constant c_2 with $f(n) \leq c_2 n^2$ for all $n > 1$. [3]
- (b) Find a constant $c_1 > 0$ with $f(n) \geq c_1 n^2$ for all $n > 1$. [2]
- (c) Hence state what you have shown, in Θ notation. [1]

4

6 marks

Use proof by contradiction, straight from the definition of O .

- (a) Show that n^2 is not $O(100n)$. [3]
- (b) Show that $2^{n+1} = O(2^n)$. [1]
- (c) Show that 2^{2n} is not $O(2^n)$. [2]

5

6 marks

Use limits to decide whether $f(n) = o(g(n))$, $f(n) = \Theta(g(n))$ or $f(n) = \omega(g(n))$.

- (a) $f(n) = n \log n$ and $g(n) = n^{1.5}$ [2]
- (b) $f(n) = 3n^2 + n$ and $g(n) = n^2$ [2]
- (c) $f(n) = 2^n$ and $g(n) = n^{10}$ [2]

6

6 marks

Here are eight functions of n :

$$n^2, \quad \log_2 n, \quad 2^n, \quad \sqrt{n}, \quad n \log_2 n, \quad \ln n, \quad 3^n, \quad 1000.$$

- (a) List them in order of asymptotic growth, slowest first. [4]
- (b) Exactly one pair grows at the same rate. Name it and explain why the two are Θ of each other. [2]

7

6 marks

Using the simplification rules, write each function as $\Theta(h(n))$ for the simplest possible h .

- (a) $7n^3 + 2n^2 \log n + 100$ [1]
- (b) $\frac{1}{4}n^2 + \frac{1}{9}n^3$ [1]
- (c) $\log_2(n^5)$ [1]
- (d) $3 \cdot 2^n + n^{100}$ [1]
- (e) $\log_3 n + \log_9 n$ [1]
- (f) $(n+1)^2 - n^2$ [1]

8

6 marks

Decide whether each statement is true or false, and justify your answer. Here f , g and h are positive functions.

- (a) If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$. [2]
- (b) If $f(n) = O(g(n))$, then $g(n) = \Omega(f(n))$. [2]
- (c) $2^n = \Theta(2^{n/2})$. [2]

9

6 marks

This question uses the c and N definition of little- o .

- (a) Show that $n = o(n^2)$: for each $c > 0$, find an N that works. [2]
- (b) Show that $3n + 1$ is not $o(n)$. [2]
- (c) Show that if $f(n) = o(g(n))$, then $f(n)$ is not $\Omega(g(n))$. [2]

10

7 marks

This question compares $\log_2(n!)$ with $n \log_2 n$.

- (a) Show that $\log_2(n!) = O(n \log_2 n)$. [2]
- (b) By keeping only the largest half of the factors in $n!$, show that $n! \geq (n/2)^{n/2}$ for $n \geq 2$. Hence show that $\log_2(n!) = \Omega(n \log_2 n)$, giving c and N . [3]
- (c) State the conclusion in Θ notation. [1]
- (d) Show that $n^{\log_2 3} = 3^{\log_2 n}$. [1]

11

8 marks

Exam style. Let $f(n) = n^3 - 10n^2 + 7$.

- (a) Find constants c_2 and N with $f(n) \leq c_2 n^3$ for all $n > N$. [2]
- (b) Find constants $c_1 > 0$ and N with $f(n) \geq c_1 n^3$ for all $n > N$. [3]
- (c) Hence state the tightest Θ bound for $f(n)$. [1]
- (d) Is $f(n) = o(n^3 \log n)$? Justify your answer with a limit. [2]

12

8 marks

Exam style. Let $f(n) = 4^{\log_2 n} + n \log_2 n$.

- (a) Simplify $4^{\log_2 n}$. [2]
- (b) Hence give the simplest $h(n)$ with $f(n) = \Theta(h(n))$. [1]
- (c) Prove your answer to (b) by giving constants c_1, c_2 and N . [3]
- (d) Is $f(n) = O(n^{2.5})$? Is $f(n) = \Omega(n^{2.5})$? Justify both. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.